Math 55: Discrete Mathematics, Fall 2008
Homework 7 Solutions

5.2: 12. 26, since there are 25 possibilities for the pair ($a \,\underline{\text{mod}}\, 5$, $b \,\underline{\text{mod}}\, 5$).

14. (a) Consider the list of 14 integers $x_1, \ldots, x_7, 11-x_1, \ldots, 11-x_7$. All 14 are between 1 and 10, so there must be at least four equal pairs, since we cannot have three integers equal on this list, and we could have at most a list of 13 with 0 to 3 equal pairs. An equal pair must be of the form $x_i = 11 - x_j$, that is, $x_i + x_j = 11$. The same x_i and x_j gives a second equal pair $x_j = 11 - x_i$, but since we have four equal pairs, we must have two solutions $x_i + x_j = 11$

(b) No. A counterexample is $\{1, \ldots, 6\}$ in which the only pair that sums to 11 is $5 + 6$.

* 36. As noted in the hint, we cannot have a person who knows no one else and a person who knows everyone else. So if $k(x)$ is the number of other people that person x knows, then all the values $k(x)$ are either in the set $\{0, 1, \ldots, n-2\}$ (if nobody knows everyone) or in $\{1, 2, \ldots, n-1\}$ (if nobody knows no one else). Either way, there are n values in a set of $n-1$ possibilities, so two must be equal.

40. Consider the 102 integers $x_1, \ldots, x_{51}, x_1 + 1, \ldots, x_{51} + 1$, where the x_i's are the 51 addresses. All these integers are in the set $[1000, 1100]$, which has 101 elements, so two most be equal. The two equal ones must be some $x_j = x_i + 1$, *i.e.*, x_i and x_j are consecutive addresses.

Ch. 5 Suppl. Ex. 18. Divide the square into four 1×1 squares. Two of the 5 points must be in the same 1×1 square, by pigeonhole. Then the distance between them is at most the length of the diagonal of the 1×1 square, which is $\sqrt{2}$.

5.3: 22(a) 7! [treat ED as if it were a single letter]

(c) 5! [treat BA and FGH as single letters]

(f) 0 [if the permutation contains BCA then B is before A and it cannot contain ABF]

* 24. $10!P(11, 6)$. The first factor counts ways to arrange the women. One they are placed, we have 11 spots (before the first woman and after each of the 10 women) in which to place the 6 men. Since we can't repeated a spot, placing the men is equivalent to choosing a 6-permutation of the 11 spots.

26. (a) $\binom{13}{10}$

(b) $P(13, 10) = 13!/3!$

(c) $\binom{13}{10} - 1$. We subtracted 1 from (a) for the unique forbidden choice in which the 10 men take the field.

38. $\binom{45}{3}\binom{57}{4}\binom{69}{5}$

5.4: 8. $\binom{17}{8}3^8 2^9$ [or equivalently, $\binom{17}{9}3^8 2^9$]

14. By symmetry $\binom{n}{k} = \binom{n}{n-k}$, it is enough to show that $\binom{n}{k} < \binom{n}{k+1}$ if $k + 1 \leq n/2$. From the formula $\binom{n}{k}$=n!/(k!(n-k)!), we find $\binom{n}{k+1}/\binom{n}{k} = (n-k)/(k+1)$. If $k + 1 \leq n/2$, then $n \geq 2k + 2$,hence $n - k > k + 1$. Therefore the ratio $(n-k)/(k+1)$ is greater than 1, showing that $\binom{n}{k} < \binom{n}{k+1}$.

* [5 each part] 22. (a) Count ways to choose an r-element subset R of an element set N, and a k-element subset K of R. Choosing R and then K gives the answer $\binom{n}{r}\binom{r}{k}$. Choosing K first, and then the $r-k$ elements of $R-K$ from $N-K$ gives $\binom{n}{k}\binom{n-k}{r-k}$.

(b) Using the formula and making appropriate cancellations, we get $n!/(k!(r-k)!(n-r)!)$ for both sides of the identity.

24. In the formula $\binom{p}{k} = p!/(k!(p-k)!)$, note that the factorial in the numerator contains the prime factor p, while for $1 \leq k \leq p-1$, the two factorials in the denominator are products of postive integers less than p, so do note contain a factor that could cancel the p in the numerator.

(A) By the Binomial Theorem $(n+1)^p = \sum_{k=0}^{p} \binom{p}{k} n^k$. By Problem 24, every term in this sum is $\equiv 0 \pmod{p}$ except those for $k=0$ and $k=p$, which are 1 and n^p, respectively. Therefore we have the identity

$$(n+1)^p \equiv n^p + 1 \pmod{p}$$

for every integer n. Using the base case $0^p \equiv 0 \pmod{p}$, and the formula above for the inductive step $(n+1)^p \equiv n^p + 1 \equiv n+1 \pmod{p}$, we prove that $n^p \equiv n \pmod{p}$ for every integer $n \geq 0$. It is also true for negative integers, because every negative integer is congruent modulo p to some positive integer.

To get Fermat's little theorem in the more familiar form $n^{p-1} \equiv 1$ for $n \not\equiv 0 \pmod{p}$, multiply both sides by the multiplicative inverse $n^{-1} \pmod{p}$, which exists since p is prime.