Math 55: Discrete Mathematics, Fall 2008
Homework 5 Solutions

4.1: 10(a) The formula is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

(b) Basis step: the formula is true for $n=1$. For $n>1$, assume by induction that the formula is true for $n-1$, so

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)n}=\frac{n-1}{n}.$$

Now add $1/(n(n+1))$ to both sides and do a little algebra on the right-hand side to see that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n-1}{n}+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

* 22. Theorem: $n^2\le n!$ for $n\ge 4$.

Proof: True for $n=4$, since $4^2=16\le 24=4!$. For $n>4$, assume by induction that $(n-1)^2\le(n-1)!$. Multiplying both sides by n gives $n(n-1)^2\le n!$. So we just need to show that $n^2\le n(n-1)^2$ for $n>4$. Since $n>4$ we have $n-n/2=n/2>2>1$, hence $n-1>n/2$, and we also have $n-1>2$. Multiplying the last two inequalities gives $(n-1)^2>n$, and multiplying this by n on both sides gives the inequality we needed to prove. There are also other ways to do the algebra for the inequality.

28. Basis step: for $n=3$, we have $n^2-7n+12=0$ so the assertion holds. We'll prove that it holds for $n+1$ if it holds for n when $n\ge 3$. The difference $((n+1)^2-7(n+1)+12)-(n^2-7n+12)$ is equal to $2n-6$. For $n\ge 3$ this is ≥ 0, and since $n^2-7n+12\ge 0$ by the induction hypothesis, we conclude that $(n+1)^2-7(n+1)+12\ge 0$.

* (A) Observe that $n^2-7n+12=(n-3)(n-4)$. For $n=3$ or $n=4$ this is zero, and for $n>4$ it is a product of two positive integers, hence positive.

48. The inductive step is OK, but the basis step is false. The formula is not true for $n=1$.

(B) If $n=1$, the assertion is tautologically true. For $n>1$, assume it holds for $n-1$, and set $a_1\cdots a_n=bc$, where $b=a_1\cdots a_{n-1}$ and $c=a_n$. By Lemma 1, p divides a_n or p divides $a_1\cdots a_{n-1}$. In the latter case, we conclude by induction that p divides one of $a_1,\dots,a_{n-1}$. So p divides some a_i in either case.

4.2: 8. The possible nonzero totals (in dollars) are 25, 40, 50, 65, 75, 80, 90, 100, 105, 115, 120, 125, 130 and every multiple of 5 greater than or equal to 140.

Proof: First note that the total must be a multiple of 5. Next verify by considering all combinations of at most five 25's and three 40's that the totals less than 140 are exactly those listed above. Now we want to prove that every total which is divisible by 5 and greater than or equal to 140 is possible.

To do this, verify directly that 140, 145, 150, 155 and 160 are possible. For $T \geq 165$ and divisible by 5, we can assume by strong induction that $T - 25$ is possible, since $T - 25$ is also divisible by 5 and $T - 25 \geq 140$. Then we can form the total T by combining $T - 25$ with another 25.

* 10. Theorem: It takes exactly $n - 1$ breaks to break the bar into its n individual squares, no matter what the shape of the rectangle.

Proof: Let the rectangle be $k \times l$, so $n = kl$, where k and l are positive integers. If $k = l = 1$, we have one square and use zero breaks, so the theorem is true in this case. Otherwise either k or l or both is > 1. Suppose $k > 1$. If we break the bar horizontally, we get new bars of size $k_1 \times l$ and $k_2 \times l$, where $k_1 + k_2 = k$, and each new bar has less than n squares. By induction, the first bar requires $k_1 l - 1$ breaks and the second requires $k_2 l - 1$ breaks. Adding these together with the one break we made initially gives $k_1 l + k_2 l - 1 = kl - 1 = n - 1$ breaks, as we were to show. The other case, when $l > 1$ and we break the bar vertically, follows by symmetry.

16. Strenghtening the induction hypothesis a bit, we will show that the first player can win at chomp on every $2 \times n$ board, and also on every 2-row board in which more than one square has already been chomped from the bottom row.

We assume by induction that the above assertion is true for all boards smaller than the one we are considering. Suppose our board is $2 \times n$. In that case, the first player should chomp one cookie at the bottom right. If $n = 1$, this clearly wins. Otherwise, the second player must either chomp from the top row, leaving a smaller $2 \times m$ board for the first player, or chomp from the bottom row, leaving a smaller board with more than one cookie chomped from the bottom row. By the inductive hypothesis, the first player (whose turn it now is again) wins.

In the other case, suppose our board is pre-chomped, with k cookies in the top row, l cookies in the bottom row, and $k - l \geq 2$ Then the first player can chomp $k - l - 1$ cookies from the top row, leaving the second player with a smaller board that looks like a $2 \times (l + 1)$ board minus one chomped cookie. As we already saw, that board is losing for the second player.