Math 55: Discrete Mathematics, Fall 2008
Homework 1 Solutions

1.1: 8(b) You pass the course if and only if you don't miss the final. 8(e) It's either the case that if you have the flu you don't pass, or if you miss the final you don't pass, or both. Note that "if you have the flu you don't pass or if you miss the final you don't pass" suggests the meaning $(p \vee q) \rightarrow \neg r$, which is not equivalent to the proposition in the problem. 8(f) Either you have the flu and miss the final, or you don't miss the final and you pass. (A) If p and q are true, then both (b) and (e) implies that r is false, *i.e.*, you don't pass, but (f) implies no conclusion about r.

12(c) The form is $p \leftrightarrow q$ where p and q are both false, so the proposition is true.

14(b) True (because 1+1=3 is false). (c) False (because the hypothesis is true and the conclusion is false).

20(a) If I remember to send you the address, then you have sent me an e-mail message. (g) If you can log on to the server, then you have a valid password. (h) If you don't begin your climb too late, then you will reach the summit.

p. 107, Ex. 10. A is a knave, since his statement would be false if he were a knight. Then B must be a knave, otherwise A's statement would be true. Finally, C must be a knave, otherwise B's statement would be true.

(B) The logical form of the statement is "if a dog barks, then it doesn't bite." The converse is "dogs that don't bite bark." The inverse is "dogs that don't bark bite." The contrapositive is "biting dogs don't bark."

1.2: 26. Using a truth table, one sees that each proposition is true in all cases except when q is true and p and r are false.

30. You can either check with a truth table, or deduce $(q \vee r)$ from the hypothesis $(p \vee q) \wedge (\neg p \vee r)$ as follows: either p is true, in which case the clause $(\neg p \vee r)$ implies that r is true, or p is false, in which case the clause $(p \vee q)$ implies that q is true. So $(q \vee r)$ is true in both cases.

40. $p \wedge q \wedge \neg r$.

1.3: 10(d) $\neg \exists x\, (C(x) \wedge D(x) \wedge F(x))$ (e) $(\exists x\, C(x)) \wedge (\exists x\, D(x)) \wedge (\exists x\, F(x))$.

16(a) True (2 has a real square root). (b) False (-1 does not have a real square root). (c) True (because x^2 is always ≥ 0). (d) False (0 and 1 are counterexamples).

60(a) $\forall x\, (P(x) \rightarrow Q(x))$. (b) $\exists x\, (R(x) \wedge \neg Q(x))$. (c) $\exists x\, (R(x) \wedge \neg P(x))$.

1.4: 24(d) The product of two real numbers is non-zero if and only if both numbers are non-zero.

28(c) True ($x = 0$ has the requisite property for all y). (d) False (the proposition asserts that addition is not commutative). (h) False (the proposition asserts that the system of equations $x + 2y = 2$, $2x + 4y = 5$ has a solution, but this system is inconsistent).

36(d) The negation, in plain English without negations left of quantifiers, reads "every student has failed to solve at least one exercise in the book."

1.5: 24. Steps 3 and 5 are erroneous because one can only simplify from a conjunction, not a disjunction. Step 7 has a wrong justification, since the proposition being formed is a disjunction; however it would be valid to deduce step 7 from either 4 or 6 by addition. (Step 7 also contains a typographical error in the placement of parentheses, but I don't think that was meant as part of the problem.)

1.3: 60(d) Yes.

1.6: 8. Suppose to the contrary that both n and $n+2$ are squares, say $n = k^2$ and $n+2 = l^2$, where k and l are integers which we may assume without loss of generality are non-negative, and hence $k < l$. Then $l^2 - k^2 = (l-k)(l+k) = 2$. Both $l-k$ and $l+k$ are non-negative integers, whose product is 2, and $l+k$ is the larger of them, so we must have $l-k = 1$, $l+k = 2$. But this has no solution in integers (the unique solution is $l = 3/2$, $k = 1/2$). [Another possible line of argument is to use the fact that $l \geq k+1$, which leads to $n+2 = l^2 \geq k^2 + 2k + 1 = n + 2k + 1$, and hence $2 \geq 2k+1$. This forces $k = 0$, hence $n = 0$, hence $n+2 = 2$, which is not a square.]

16. By contraposition, it is equivalent to prove that if m and n are both odd, then so is mn. So assume $m = 2k+1$, $n = 2l+1$. Then $mn = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$, which is odd.

28. The "if" part is obvious. For the "only if" part, suppose $m^2 = n^2$. Then $m^2 - n^2 = (m+n)(m-n) = 0$. One of the factors $m+n$ or $m-n$ must therefore be equal to 0, yielding $m = -n$ or $m = n$.

1.7: 4. Consider six cases corresponding to six possible orderings $a \leq b \leq c$, $a \leq c \leq b$, and so on, and calculate both sides of the equation in each case. [Another possible strategy, with fewer cases but a more complicated proof for each case, is to consider three cases, according to which of a, b and c is the smallest of the three numbers.]

22. We'll prove that the quadratic mean is greater than or equal to the arithmetic mean. We begin with the fact that $(x-y)^2 = x^2 - 2xy + y^2 \geq 0$. Add $x^2 + 2xy + y^2 = (x+y)^2$ to the two sides of this inequality to get $2x^2 + 2y^2 \geq (x+y)^2$. Now divide both sides by 4 and take square roots to get $\sqrt{(x^2+y^2)/2} \geq |(x+y)/2| \geq (x+y)/2$. [The hard part of constructing this proof is of course to discover the above line of algebraic reasoning.]

24. To reach a final state of 9 0's, the previous state must have all bits equal, thus it must be either 9 0's or 9 1's. To reach a state of 9 1's, the previous state must alternate 0's and 1's all the way around the circle, which is impossible since 9 is odd. So the only way we can ever reach 9 0's is either to start with 9 0's, or start with 9 1's. In particular, if we start with 5 1's and 4 0's, we can never reach 9 0's.

2.1: 36(a) The positive integers. (b) The empty set. (c) $\mathbb{Z} - \{0, 1\}$.

2.2: 36. We must show that each of the sets $A \oplus B$ and $(A-B) \cup (B-A)$ contains the other. Suppose $x \in A \oplus B$. By definition, $x \in A \cup B$, so consider two cases. If $x \in A$, then, since x is required not to be in $A \cap B$, we must have $x \notin B$; thus $x \in (A-B)$. In the other case, $x \in B$, we see similarly that $x \notin A$, thus $x \in (B-A)$. In either case we have $x \in (A-B) \cup (B-A)$. For the reverse containment, suppose $x \in (A-B) \cup (B-A)$. Since this set is a union, we again consider two cases. First, if $x \in (A-B)$, then $x \in A$, hence $x \in A \cup B$, but $x \notin B$, hence $x \notin A \cup B$. Therefore $x \in A \oplus B$ by the definition of the latter set. Exchanging the roles of A and B we see that in the other case, $x \in (B - A)$, we again have $x \in A \oplus B$.