8. Let $G$ be an algebraic group over $k$, $\text{Der}(O(G))$ the space of $k$-derivations $\xi : O(G) \to O(G)$, and $\text{Der}^G(O(G))$ the space of left-invariant derivations (i.e., $\xi$ commutes with the action of $G$ on $O(G)$ corresponding to the action of $G$ on itself by left multiplication). Let $\text{Der}_e(O(G), k)$ be the space of $e$-derivations $\xi : O(G) \to k$, where $\epsilon : O(G) \to k$ is the counit (the algebra homomorphism given by evaluation at $1 \in G$). Let $O(G)^o$ be the Hopf dual and $O(G)^o_{\text{pr}}$ be its subspace of primitive elements (i.e., $x$ such that $\Delta x = x \otimes 1 + 1 \otimes x$).

(a) Verify that the commutator of two derivations is a derivation, giving $\text{Der}(O(G))$ the structure of a Lie algebra.

(b) Verify that $\text{Der}^G(O(G))$ is a Lie subalgebra of $\text{Der}(O(G))$.

(c) Verify that the natural map $\text{Der}^G(O(G)) \to \text{Der}_e(O(G))$ given by evaluation at $1$ is a linear isomorphism.

(d) Verify that the primitive elements of any Hopf algebra form a Lie algebra, with the bracket given by commutator in that Hopf algebra.

(e) Verify that the inclusion $\text{Der}_e(O(G)) \subseteq O(G)^*$ maps $\text{Der}_e(O(G))$ isomorphically onto $O(G)^o_{\text{pr}}$.

(f) Via the isomorphisms in (c) and (e), the space $\text{Der}_e(O(G))$ acquires two Lie algebra structures. Show that they are the same. (Geometrically, $\text{Der}_e(O(G))$ is the tangent space to $G$ at $1$. The Lie algebra structure given by (c) is the one usually taken as the definition, although (e) is more natural from a Hopf algebra point of view.)

9. Fix a basis $e_1, \ldots, e_{2n}$ of $k^{2n}$ and let $\langle -, - \rangle$ be the antisymmetric form such that $\langle e_i, e_{2n+1-i} \rangle = 1$ for $i = 1, \ldots, n$, and $\langle e_i, e_j \rangle = 0$ if $j \neq 2n+1-i$. (In other words, the matrix $J$ of the form is antidiagonal with $1$'s in the upper half and $-1$'s in the lower half).

The symplectic group $Sp_{2n}(k)$ is the subgroup of $GL_{2n}(k)$ consisting of elements that preserve the symplectic form $\langle -, - \rangle$. The upper triangular matrices in $Sp_{2n}(k)$ form a Borel subgroup $B$, and the diagonal matrices for a maximal torus $T \subseteq B$.

(a) Show that the Lie algebra $sp_{2n}$ of $Sp_{2n}$ consists of matrices of the block form

$$\begin{bmatrix} A & B \\ C & -A^R \end{bmatrix},$$

where $A^R$ denotes the transpose of $A$ about the antidiagonal.

(b) Describe the character lattice $X = X(T)$, cocharacter lattice $Y$, roots, coroots, simple roots and simple coroots, root lattice $Q$ and coroot lattice $Q^\vee$. Determine the Cartan matrix and the corresponding Dynkin diagram.

(c) Is $Sp_{2n}$ simply connected? Is it adjoint? Describe all the reductive algebraic groups isogenous to $Sp_{2n}$ (i.e., they have the same Lie algebra).

10. Show that the even orthogonal groups $SO_{2n}$ are neither adjoint nor simply connected. Show that the corresponding adjoint group is $SO_{2n}/\{\pm 1\}$. The simply connected cover is called $\text{Spin}_{2n}$. Show that if $n$ is odd, then $SO_{2n}$ is the only intermediate group between $\text{Spin}_{2n}$ and $SO_{2n}/\{\pm 1\}$, but if $n$ is even, there are two others, isomorphic to each other via an isomorphism that induces a nontrivial automorphism of the Lie algebra.
11. Let $V$ be a representation of an algebraic group, or of a quantum group, which has a decomposition $V = \bigoplus_{\lambda \in X} V_\lambda$ into finite-dimensional weight spaces. Define the character $\chi_V$ to be the formal sum $\sum_{\lambda} \dim(V_\lambda) e^\lambda$ in $k[X]$, where $I$ denote the image in $k[X]$ of $\lambda \in X$ by $e^\lambda$ to stress that the additive group law in $X$ is written as multiplication in $k[X]$, that is, $e^{\lambda + \mu} = e^\lambda e^\mu$. Prove that if $V$ is integrable then $\chi_V$ is invariant under the action of the Weyl group on $k[X]$ induced by its action on $X$.

(In the Kac-Moody case, $V$ might be infinite-dimensional but everything still makes sense as long as it has finite-dimensional weight spaces.)

12. Prove that the only possible coproducts on $U_q(sl_2)$ of the form $\Delta E = E \otimes a(K) + b(K) \otimes E$, $\Delta F = F \otimes c(K) + d(K) \otimes F$ are the usual $\Delta$, the coproduct $\bar{\Delta}$ obtained by interchanging $K$ and $K^{-1}$ in the usual one, and their opposites $\Delta^{op}$ and $\bar{\Delta}^{op}$.

13. Let $L_m$ denote the irreducible representation of $U_q(sl_2)$ with highest weight $m$ (i.e., $K$ acts as $q^m$). Show that the decomposition into irreducibles of tensor products is given by

$$L_m \otimes L_n \cong L_{|m-n|} \oplus L_{|m-n|+2} \oplus \cdots \oplus L_{m+n-2} \oplus L_{m+n}.$$

(This is easy, using characters.)

14. Let $\epsilon$ be a primitive $l$-th root of unity, where $l > 1$ is odd (so $\epsilon^2$ is also a primitive $l$-th root of unity). We can define an algebra $U_{\epsilon}(sl_2)$ with the same generators and relations as for $U_q(sl_2)$ but with $q$ replaced by $\epsilon$.

(a) Show that for any numbers $\alpha, \beta$ and $\gamma \neq 0$, this algebra has an $l$-dimensional module $L_{\alpha,\beta,\gamma}$ on which $e^l$ and $f^l$ act as the scalars $\alpha$ and $\beta$, and $K$ acts with eigenvalues $\gamma, \epsilon^{2\gamma}, \ldots, \epsilon^{2l-2\gamma}$.

(b) Show that $L_{\alpha,\beta,\gamma}$ is irreducible unless $\alpha = \beta = 0$ and $\gamma^{2l} = 1$.

15. Let $U = U_q(sl_2)$, and recall that the element $\Theta \in (U_- \otimes U_+)$ such that $\Theta \bar{\Delta}(x) = \Delta(x)\Theta$ for all $x \in U$ is given by

$$\Theta = \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} (q - q^{-1})^n [n]_q ! f^{(n)} \otimes e^{(n)}.$$ 

Show that $\Theta = \bar{T} T^{-1}$, where

$$T = \sum_{n=0}^{\infty} q^{n^2} f^{(n)} \otimes e^{(n)},$$

and $\bar{T}$ is the same with $q$ replaced by $q^{-1}$. 