Math 274—Quantum Groups—Fall 2004

More exercises

8. Let G be an algebraic group over k, $\mathrm{Der}(\mathcal{O}(G))$ the space of k-derivations $\xi\colon \mathcal{O}(G) \to \mathcal{O}(G)$, and $\mathrm{Der}^G(\mathcal{O}(G))$ the space of left-invariant derivations (*i.e.*, ξ commutes with the action of G on $\mathcal{O}(G)$ corresponding to the action of G on itself by left multiplication). Let $\mathrm{Der}_\epsilon(\mathcal{O}(G), k)$ be the space of ϵ-derivations $\zeta\colon \mathcal{O}(G) \to k$, where $\epsilon\colon \mathcal{O}(G) \to k$ is the counit (the algebra homomorphism given by evalueation at $1 \in G$). Let $\mathcal{O}(G)^\circ$ be the Hopf dual and $\mathcal{O}(G)^\circ_{\mathrm{pr}}$ be its subspace of primitive elements (*i.e.*, x such that $\Delta x = x \otimes 1 + 1 \otimes x$).

(a) Verify that the commutator of two derivations is a derivation, giving $\mathrm{Der}(\mathcal{O}(G))$ the structure of a Lie algebra.

(b) Verify that $\mathrm{Der}^G(\mathcal{O}(G))$ is a Lie subalgebra of $\mathrm{Der}(\mathcal{O}(G))$.

(c) Verify that the natural map $\mathrm{Der}^G(\mathcal{O}(G)) \to \mathrm{Der}_\epsilon(\mathcal{O}(G))$ given by evaluation at 1 is a linear isomorphism.

(d) Verify that the primitive elements of any Hopf algebra form a Lie algebra, with the bracket given by commutator in that Hopf algebra.

(e) Verify that the inclusion $\mathrm{Der}_\epsilon(\mathcal{O}(G)) \subseteq \mathcal{O}(G)^*$ maps $\mathrm{Der}_\epsilon(\mathcal{O}(G))$ isomorphically onto $\mathcal{O}(G)^\circ_{\mathrm{pr}}$.

(f) Via the isomorphisms in (c) and (e), the space $\mathrm{Der}_\epsilon(\mathcal{O}(G))$ acquires two Lie algebra structures. Show that they are the same. (Geometrically, $\mathrm{Der}_\epsilon(\mathcal{O}(G))$ is the tangent space to G at 1. The Lie algebra structure given by (c) is the one usually taken as the definition, although (e) is more natural from a Hopf algebra point of view.)

9. Fix a basis $e_1, \ldots, e_{2n}$ of k^{2n} and let $\langle -,- \rangle$ be the antisymmetric form such that $\langle e_i, e_{2n+1-i} \rangle = 1$ for $i = 1, \ldots, n$, and $\langle e_i, e_j \rangle = 0$ if $j \neq 2n+1-i$. (In other words, the matrix J of the form is antidiagonal with 1's in the upper half and -1's in the lower half).

The *symplectic group* $Sp_{2n}(k)$ is the subgroup of $GL_{2n}(k)$ consisting of elements that preserve the symplectic form $\langle -,- \rangle$. The upper triangular matrices in $Sp_{2n}(k)$ form a Borel subgroup B, and the diagonal matrices for a maximal torus $T \subseteq B$.

(a) Show that the Lie algebra $\mathfrak{sp}_{2n}$ of Sp_{2n} consists of matrices of the block form

$$\begin{bmatrix} A & B = B^R \\ C = C^R & -A^R \end{bmatrix},$$

where A^R denotes the transpose of A about the antidiagonal.

(b) Describe the character lattice $X = X(T)$, cocharacter lattice Y, roots, coroots, simple roots and simple coroots, root lattice Q and coroot lattice $Q^\vee$. Determine the Cartan matrix and the corresponding Dynkin diagram.

(c) Is Sp_{2n} simply connected? Is it adjoint? Describe all the reductive algebraic groups isogenous to Sp_{2n} (*i.e.*, they have the same Lie algebra).

10. Show that the even orthogonal groups SO_{2n} are neither adjoint nor simply connected. Show that the corresponding adjoint group is $SO_{2n}/\{\pm 1\}$. The simply connected cover is called Spin_{2n}. Show that if n is odd, then SO_{2n} is the only intermediate group between Spin_{2n} and $SO_{2n}/\{\pm 1\}$, but if n is even, there are two others, isomorphic to each other via an isomorphism that induces a nontrivial automorphism of the Lie algebra.

11. Let V be a representation of an algebraic group, or of a quantum group, which has a decomposition $V = \bigoplus_{\lambda \in X} V_\lambda$ into finite-dimensional weight spaces. Define the *character* χ_V to be the formal sum $\sum_\lambda \dim(V_\lambda) e^\lambda$ in $k[X]$, where I denote the image in $k[X]$ of $\lambda \in X$ by e^λ to stress that the additive group law in X is written as multiplication in $k[X]$, that is, $e^{\lambda+\mu} = e^\lambda e^\mu$. Prove that if V is integrable then χ_V is invariant under the action of the Weyl group on $k[X]$ induced by its action on X.

(In the Kac-Moody case, V might be infinite-dimensional but everything still makes sense as long as it has finite-dimensional weight spaces.)

12. Prove that the only possible coproducts on $U_q(\mathfrak{sl}_2)$ of the form $\Delta E = E \otimes a(K) + b(K) \otimes E$, $\Delta F = F \otimes c(K) + d(K) \otimes F$ are the usual Δ, the coproduct $\overline{\Delta}$ obtained by interchanging K and K^{-1} in the usual one, and their opposites Δ^{op} and $\overline{\Delta}^{\mathrm{op}}$.

13. Let L_m denote the irreducible representation of $U_q(\mathfrak{sl}_2)$ with highest weight m (*i.e.*, K acts as q^m). Show that the decomposition into irreducibles of tensor products is given by

$$L_m \otimes L_n \cong L_{|m-n|} \oplus L_{|m-n|+2} \oplus \cdots \oplus L_{m+n-2} \oplus L_{m+n}.$$

(This is easy, using characters.)

14. Let ϵ be a primitive l-th root of unity, where $l > 1$ is odd (so ϵ^2 is also a primitive l-th root of unity). We can define an algebra $U_\epsilon(\mathfrak{sl}_2)$ with the same generators and relations as for $U_q(\mathfrak{sl}_2)$ but with q replaced by ϵ.

(a) Show that for any numbers α, β and $\gamma \neq 0$, this algebra has an l-dimensional module $L_{\alpha,\beta,\gamma}$ on which e^l and f^l act as the scalars α and β, and K acts with eigenvalues $\gamma, \epsilon^2\gamma, \ldots, \epsilon^{2l-2}\gamma$.

(b) Show that $L_{\alpha,\beta,\gamma}$ is irreducible unless $\alpha = \beta = 0$ and $\gamma^{2l} = 1$.

15. Let $U = U_q(\mathfrak{sl}_2)$, and recall that the element $\Theta \in (U_- \otimes U_+)\hat{}$ such that $\Theta\overline{\Delta}(x) = \Delta(x)\Theta$ for all $x \in U$ is given by

$$\Theta = \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} (q - q^{-1})^n [n]_q! f^{(n)} \otimes e^{(n)}.$$

Show that $\Theta = \overline{T} T^{-1}$, where

$$T = \sum_{n=0}^{\infty} q^{n^2} f^{(n)} \otimes e^{(n)},$$

and $\overline{T}$ is the same with q replaced by q^{-1}.