Math 274—Quantum Groups—Fall 2004

Exercises

1. Let $T = (\mathbb{G}_m)^n$ be an algebraic torus and $X = X(T)$ its lattice of characters $\chi\colon T \to \mathbb{G}_m$. Since $\mathbb{G}_m = k^* \subseteq k$, every character is in particular a function $\chi \in \mathcal{O}(T)$. Show that this tautological embedding of X into $\mathcal{O}(T)$ induces an isomorphism of Hopf algebras $kX \cong \mathcal{O}(T)$. Here kX is the group algebra of X, with the usual Hopf algebra structure in which $\Delta(\chi) = \chi \otimes \chi$ for $\chi \in X$.

2. If G is a finite group, show that $\mathcal{O}(G)^\circ = kG$ and vice versa.

3. Let G be an algebraic group. An element $g \in \mathcal{O}(G)^\circ$ is called *grouplike* if $\Delta(g) = g \otimes g$. Prove that the grouplike elements are exactly the evaluation homomorphisms ε_g for $g \in G$, and they form a basis of a sub-Hopf-algebra of $\mathcal{O}(G)^\circ$ which is isomorphic to kG.

4. Let a group G act on an algebra A by algebra automorphisms.
(a) Define a "twisted" product on $A \otimes kG$ by

$$(a \otimes g)(b \otimes h) = a\, g(b) \otimes gh.$$

Verify that this makes $A \otimes kG$ an associative algebra, generated by subalgebras $A \otimes 1 \cong A$ and $1 \otimes kG \cong kG$, such that $ga = g(a)g$ for $g \in G$, $a \in A$.
(b) Suppose that A is a bialgebra and that G acts by bialgeba automorphisms. Give $A \otimes kG$ the usual coalgebra structure induced from those on A and kG. Show that $A \otimes kG$ (with the twisted product above) is a bialgebra; and if A has an antipode, then so does $A \otimes kG$.

5. Let G be an algebraic group and let $\mathfrak{m} = \ker(\varepsilon)$ be the maximal ideal of the unit element $1 \in G$. As in the lecture, consider the subalgebra $\mathcal{O}(G)^\circ_1 = \{\zeta \in \mathcal{O}(G)^\circ : \exists N\ \zeta|_{\mathfrak{m}^N} = 0\}$. The action of G on itself by conjugation induces an action of G on $\mathcal{O}(G)^\circ_1$. Prove that $\mathcal{O}(G)^\circ$ is isomorphic as a bialgebra to $\mathcal{O}(G)^\circ_1 \otimes kG$, where the latter is given the twisted product and usual coproduct as in Exercise 4.

5. (a) Show that the following two properties of an algebraic group G are equivalent: (i) every representation G of V has a basis in which G acts by upper unit triangular matrices, (ii) the only irreducible representation of G is the one-dimensional trivial representation. (Either property can be taken as the definition of *unipotent*.)
(b) Show that if N is a normal subgroup of G and both N and G/N are unipotent, then G is unipotent.
(c) Show that if G has a faithful representation as a group of upper unit triangular matrices, then G is unipotent.

6. If (A, Δ, ε) is a coalebra with counit, and R is an algebra with unit, denote multiplication in R by $m_R\colon R \otimes R \to R$, and define the *convolution product*

$$f \cdot g = m_R \circ (f \otimes g) \circ \Delta$$

on $\mathrm{Hom}_k(A, R)$
(a) Show that the convolution product makes $\mathrm{Hom}_k(A, R)$ into an associative algebra with unit element ε (where we consider ε as a map $A \to R$ via $k \hookrightarrow R$).

(b) Assume from now on that A is a bialgebra. Prove that $S \in \mathrm{Hom}_k(A, A)$ is an antipode if and only if S is a two-sided inverse to the identity map $\mathrm{id}_A \in \mathrm{Hom}_k(A, A)$ with respect to the convolution product. Deduce that an antipode is unique if it exists.

(c) Show that $A \otimes A$ is a bialgebra with the usual product, and coproduct defined by

$$\Delta(a \otimes b) = (a^{(1)} \otimes b^{(1)}) \otimes (a^{(2)} \otimes b^{(2)})$$

(using Sweedler notation with summation convention), and that $A \otimes A$ has counit $\varepsilon(a \otimes b) = \varepsilon(a)\varepsilon(b)$.

(d) Let $m^{\mathrm{op}} \colon A \otimes A \to A$ be the opposite multiplication, $m^{\mathrm{op}}(a \otimes b) = ba$. If A has an antipode S, prove that $S \circ m^{\mathrm{op}}$ is a 2-sided convolution inverse to m^{op} in $\mathrm{Hom}_k(A \otimes A, A)$.

(e) Show that in the situation of part (d), $m \circ (S \otimes S)$ is also a 2-sided convolution inverse to m^{op}. Deduce that an antipode S is necessarily an antihomomorphism, *i.e.* the identity $S(ab) = S(b)S(a)$ holds.

(f) Show that an antipode is necessarily also an antihomomorphism with respect to the coproduct, *i.e.*, $\Delta(S(a)) = S(a^{(2)}) \circ S(a^{(1)})$ in Sweedler notation.

7. If A is a bialgebra, and M, N are A-modules, then $M \otimes N$ (which is naturally an $A \otimes A$ module) becomes an A-module via $\Delta \colon A \to A \otimes A$. Similarly, k becomes an A-module via the counit $\varepsilon \colon A \to k$, and in view of Exercise 6(d), $\mathrm{Hom}_k(M, N)$ becomes an A-module via $(\mathrm{id}_A \otimes S) \circ \Delta \colon A \to A \otimes A^{\mathrm{op}}$.

(a) Verify that the natural isomorphisms of vector spaces $M \otimes (N \otimes Q) \cong (M \otimes N) \otimes Q$, $M \otimes k \cong M \cong k \otimes M$, and $\mathrm{Hom}_k(M \otimes N, Q) \cong \mathrm{Hom}_k(M, \mathrm{Hom}_k(N, Q))$ become A-module homomorphisms.

(b) Give a counterexample to show that if A is not co-commutative, then in general the natural isomorphism of vector spaces $M \otimes N \cong N \otimes M$ is not an A-module homomorphism.