Math 261B—Fall 2020
Problem Set 2

13. Fix a non-degenerate antisymmetric form $\langle \cdot, \cdot \rangle$ on K^{2n}, so the symplectic group $Sp(2n, K)$ is the subgroup of $GL_{2n}(K)$ fixing the form. Let $GSp(2n)$ be the subgroup of elements $g \in GL_{2n}(K)$ fixing the form up to a similarity $\langle gx, gy \rangle = \lambda(g)\langle x, y \rangle$.

(a) Show that λ is a homomomorphism $GSp(2n) \to \mathbb{G}_m$, with kernel $Sp(2n)$.

(b) Show that $GSp(2n)$ can also be described as the subgroup of GL_{2n} generated by $Sp(2n)$ and scalar multiples of the identity matrix. Assuming $\mathrm{char}(K) \neq 2$, show that this identifies $GSp(2n)$ with the quotient of $\mathbb{G}_m \times Sp(2n)$ by a central $\mathbb{Z}/2\mathbb{Z}$, with the similitude character λ given by the square on $\mathbb{G}_m$ and trivial on $Sp(2n)$.

(c) Describe the Cartan data of weight lattice X, roots and coroots, and simple roots and coroots for $GSp(2n)$.

14. Repeat the previous problem for the group $GSO(2n)$, suitably defined.

15. The compact real form of a complex reductive Lie group G is the (unique) real Lie subgroup $G_{\mathbb{R}}$ such that the Lie algebra of G is $\mathbb{C}$ tensored over $\mathbb{R}$ with the Lie algebra of $G_{\mathbb{R}}$.

Show that the compact real forms of $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ are the unitary and special unitary groups $U_n(\mathbb{C})$ and $SU_n(\mathbb{C})$.

16. Show that the compact real form of the symplectic group $Sp(2n, \mathbb{C})$ is its intersection with the unitary group $U(2n)$ inside $GL(2n, \mathbb{C})$, when the symplectic and Hermitian forms preserved by these groups are suitably chosen.

17. Show that the compact real form of $SO(N, \mathbb{C})$ is the real orthogonal group $SO(N, \mathbb{R})$, preserving a positive definite symmetric form.

18. Recall that the spin group Spin_{2n} is the simply connected covering group of $SO(2n)$, that is, the group with the same root system except that the coweight lattice is equal to the coroot lattice. Show that if n is odd, the center of Spin_{2n} is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, hence its only quotient groups are $\mathrm{Spin}_{2n} \to SO(2n) \to SO(2n)/(\pm I)$. Show that if n is even, then center of Spin_{2n} is $(\mathbb{Z}/2\mathbb{Z})^2$, giving two other quotient groups in addition to these.

19. Show that a non-degenerate antisymmetric form on K^N exists if and only if N is even (assuming $\mathrm{char}\, K \neq 2$) and is unique up to change of basis (for N even, and any K).

20. With $O(N)$ defined as the subgroup of $GL(N)$ that preserves a standard quadratic form, work out its equations and its Lie algebra, and verify that the Lie algebra reduces to dimension $\binom{N}{2}$, in any characteristic, and indeed to a free $\mathbb{Z}$ module of rank $\binom{N}{2}$ for $O(N)$ defined over $\mathbb{Z}$.

21. Given a non-degenerate antisymmetric form on K^N, we say that a subspace $V \subseteq K^N$ is *isotropic* if the form restricts to zero on $V \times V$. Given a standard quadratic form Q, we say that V is isotropic if $Q(v) = 0$ for all $v \in V$. Show that the flag varieties G/B (where B is a Borel subgroup) for $G = Sp(2n)$ and $G = SO(2n+1)$ can be identified with the spaces of maximal flags of isotropic subspaces $V_1 \subset \cdots \subset V_n$ of dimensions $\dim(V_d) = d$. For

$SO(2n)$, show that each almost maximal flag $V_1 \subset \cdots \subset V_{n-1}$ extends to two maximal flags, and that G/B can be identified with the space of almost maximal flags.

22. For $\mathfrak{U}_v(\mathfrak{b})$ and $\mathcal{O}_v(B)$ associated to the Borel subgroup $B \subseteq SL_2$, as in Lecture 23, show that the elements $K^a E^m$ and $t^a x^m$, respectively, with $a \in \mathbb{Z}$ and $m \in \mathbb{N}$, are bases of the two algebras. Compute the Hopf pairing $\langle K^a E^m, t^b x^n \rangle$ on these bases.