Math 261B—Fall 2020
Problem Set 1

1. (a) Show that the polynomial ring $A = \mathbb{Z}[x_1, \ldots, x_n]$ has a unique structure of Hopf algebra over $\mathbb{Z}$ such that the variables x_i are primitive, meaning that $\Delta x_i = x_1 \otimes 1 + 1 \otimes x_i$, the antipode is $S(x_i) = -x_i$, and the co-unit is $\varepsilon(f) = f(0)$.

(b) If K is an algebraically closed field, $K \otimes_{\mathbb{Z}} A$ becomes the coordinate ring of the additive group $\mathbb{G}_a^n(K)$. Show that this is a special case of the more general property that for any commutative ring R, the Hopf algebra structure on A gives the set of ring homomorphisms $A \to R$ the structure of a group isomorphic to $(R^n, +)$.

(c) Verify that the group structure in part (b) is functorial with respect to ring homomorphisms $R \to R'$.

In the language of algebraic geometry, there is an affine scheme $\operatorname{Spec}(A)$ over $\mathbb{Z}$ associated to A. Any scheme X over a commutative ring k induces a functor $X(-)$ from k-algebras R to sets, where $X(R)$ is the set of k-scheme morphisms $\operatorname{Spec}(R) \to X$, also called the set of R-valued points of X. Morphisms of affine k-schemes $\operatorname{Spec}(R) \to \operatorname{Spec}(A)$ are in functorial one-to-one correspondence with k-algebra homomorphisms $A \to R$. Note that every commutative ring is a $\mathbb{Z}$-algebra in a unique way.

What this problem shows is that for A as in part (a), $\operatorname{Spec}(A)$ is a group scheme $\mathbb{G}_a^n$ over $\mathbb{Z}$ whose functor of points send R to the additive group R^n.

2. Work out the analog of Problem 1 for the Laurent polynomial ring $A = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, with each variable grouplike, meaning that $\Delta x_i = x_i \otimes x_i$, and with antipode $S(x_i) = x_i^{-1}$ and co-unit $\varepsilon(f) = f(1)$.

3. The axioms of a non-commutative Hopf algebra (over a commutative ground ring k) are essentially the same as in the commutative case, except that the antipode S is required to be an a k-algebra antihomomorphism (i.e., it reverses multiplication). The coproduct $\Delta \colon A \to A \otimes A$ and co-unit $\varepsilon \colon A \to k$ are ordinary algebra homomorphisms.

(a) Write down the axioms explicitly, using Sweedler notation $\Delta f = \sum f_{(1)} \otimes f_{(2)}$ for the coproduct, recalling that they should formally make A a group object in the opposite of the category of k-algebras, with $\otimes$ playing the role of a product.

(b) Show that, even in the commutative case, the antipode is a homomorphism from the coproduct to the opposite coproduct.

(c) Show that the axioms are self-dual. Hence, in particular, if k is a field and A is a finite-dimensional Hopf algebra, then A^* has the structure of a Hopf algebra.

(d) Prove that if the antipode S in a Hopf algebra A is invertible (this holds automatically in some cases, including commutative and co-commutative Hopf algebras), then S^{-1} is the antipode for the same algebra with opposite coproduct, and also the antipode for the algebra with the same coproduct and the opposite product. In particular, if A is either commutative or co-commutative, then S^2 is the identity.

4. Show that the group algebra kG of a finite group G has a co-commutative Hopf algebra structure in which the elements of g are grouplike, and that this makes kG dual to the Hopf algebra of k-valued functions on G. This works for any commutative ring k.

5. If A is an infinite-dimensional Hopf algebra over a field k, its dual space A^* becomes an algebra, but not a co-algebra, because $\Delta^*\colon A^* \to (A \otimes A)^*$ need not map A^* into $A^* \otimes A^*$, which is a proper subspace of $(A \otimes A)^*$.

Let $A^\circ \subseteq A^*$ be the preimage $(\Delta^*)^{-1}(A^* \otimes A^*)$. Show that A° is a subalgebra of A^* and that the Hopf algebra structure on A induces a dual Hopf algebra structure on A°. This alebra is called the Hopf dual of A.

6. Show that the free abelian group generated by the elements $x^n/n!$ in $\mathbb{Q}[x]$ is a $\mathbb{Z}$-subalgebra $D \subseteq \mathbb{Q}[x]$, called the divided power algebra in one variable. Show that D has a natural co-commutative Hopf algebra structure, dual to the Hopf algebra structure on $\mathbb{Z}[x]$ in which x is primitive. Duality in this case means a perfect pairing $D \otimes_{\mathbb{Z}} \mathbb{Z}[x] \to \mathbb{Z}$ such that the coproduct, co-unit and antipode in each algebra are dual to the product, unit and antipode in the other.

7. Consider the case of the pair of dual Hopf algebras kG and $\mathcal{O}(G)$ in Problem 4, where $\mathcal{O}(G)$ is the algebra of functions $G \to k$, when k is a field of characteristic p and $G = \mathbb{Z}/p\mathbb{Z}$ is a cyclic group of order p.

(a) Show that $k(G) = k[x]/(x^p - 1)$, with x group-like. We can think of it as $\mathcal{O}(\mu_p)$ for the non-reduced group scheme of 'p-th roots of unity' over k.

(b) Show that linear representations of G over k, or kG modules, are the same as $\mathcal{O}(G)$ comodules, and that except for the fact that G is not connected group, it behaves just like a unipotent linear algebraic group, in the equivalent senses that (i) the only irreducibe kG module is the trivial representation; (ii) in every finite-dimensional representation, G acts by upper unitriangular matrices in some basis; (iii) G acts unipotently on $\mathcal{O}(G)$.

(c) Show that $\mathcal{O}(G)$ modules, or kG comodules, are the same as $(\mathbb{Z}/p\mathbb{Z})$-graded vector spaces. In particular, $\mathcal{O}(G)$ has p distinct non-isomorphic one-dimensional irreducible modules, and every module is a direct sum of these. In this sense the group scheme μ_p is 'reductive' and its representation theory in any characteristic resembles the characteristic zero representation theory of a cyclic group of order p.

8. (a) Write out explicitly the axioms of a right coaction $W \to W \otimes A$, where A is a Hopf algebra over k and W is a k module, dual to the axioms of a group action. You can assume k is a field if you like, but the axioms are the same over any commutative ring.

(b) Verify in detail that if $G \times V \to V$ is a linear algebraic action of an affine algebraic group on a finite dimensional vector space V (considered as an algebraic variety), and $\rho\colon \mathcal{O}(V) \to \mathcal{O}(V) \otimes \mathcal{O}(G)$ is the corresponding homomorphism of algebras of functions, then ρ maps V^* into $V^* \otimes \mathcal{O}(G)$ and makes V^* a $\mathcal{O}(G)$ comodule. Note that V^* is the subspace of $\mathcal{O}(V)$ consisting of linear functions.

(c) Verify in detail that every finite-dimensional right $\mathcal{O}(G)$ comodule W arises from a unique linear algebraic action of G on $V = W^*$ as in part (b), and more explicitly, that the right action of $g \in G$ on W dual to the left action on V is given by composing $\rho\colon W \to W \otimes \mathcal{O}(G)$ with the evaluation map $\mathrm{ev}_g\colon \mathcal{O}(G) \to k$.

9. (a) Show that if G is an algebraic group, the grouplike elements of the Hopf algebra $\mathcal{O}(G)$ are the 1-dimensional characters of G, that is, the group homomorphisms

$G \to \mathbb{G}_m(K) = K^\times$, considered as functions on G.

(b) More generally, show that if A is a Hopf algebra over a commutative ring k, then grouplike elements of A correspond naturally to A comodules isomorphic to k as a k module.

10. Let A be a Hopf algebra over k. Show that a k-linear map $\lambda\colon A \to k$ is a grouplike element of the Hopf dual of A if and only if is an algebra homomorphism.

In the case $A = \mathcal{O}(G)$ for an affine algebraic group G, this means that the grouplike elements of the Hopf dual of A correspond to the group elements $g \in G$.

11. Let G be an affine algebraic group and let $A = \mathcal{O}(G)$. Recall that G embeds in the the algebra A^* by $g \to \mathrm{ev}_g$ and that its Lie algebra $\mathfrak{g} = T_e G$ embeds in A^* as the space of linear functionals that kill the ideal $\mathfrak{m}_e^2$ and the constant functions. Verify in detail that the adjoint action of G on $\mathfrak{g}$, given for $g \in G$ by the diffential at e of conjugation by g on G, corresponds to conjugation by G on $\mathfrak{g}$ in A^*.

12. Show that in characteristic zero, the finite dimensional representations of $GL_2(K)$ are completely reducible, and the irreducible representations are the standard representations on homogeneous polyonomials of each degree d in $K[x, y]$, tensored with integer powers of the 1-dimensional representation whose character is the determinant. More precisely, with $T \subset B \subset GL_2$ the diagonal and upper triangular matrices, the weight lattice is $X = \mathbb{Z}^2$, the dominant weights are (λ_1, λ_2) such that $\lambda_1 \geq \lambda_2$, and $V_\lambda = (\det)^{\otimes \lambda_2} \otimes K[x, y]_{\lambda_1 - \lambda_2}$ is the irreducible representation with highest weight λ.

More explicitly, show that the coordinate ring $\mathcal{O}(GL_2)$ decomposes into a direct sum of subspaces spanned by the matrix coefficients of the representations described above. This and the irreducibility of these represenations implies the other conclusions.