1. (a) Show that the polynomial ring \( A = \mathbb{Z}[x_1, \ldots, x_n] \) has a unique structure of Hopf algebra over \( \mathbb{Z} \) such that the variables \( x_i \) are primitive, meaning that \( \Delta x_i = x_1 \otimes 1 + 1 \otimes x_i \), the antipode is \( S(x_i) = -x_i \), and the co-unit is \( \varepsilon(f) = f(0) \).

(b) If \( K \) is an algebraically closed field, \( K \otimes_{\mathbb{Z}} A \) becomes the coordinate ring of the additive group \( \mathbb{G}_m(K) \). Show that this is a special case of the more general property that for any commutative ring \( R \), the Hopf algebra structure on \( A \) gives the set of ring homomorphisms \( A \rightarrow R \) the structure of a group isomorphic to \( (R^n, +) \).

(c) Verify that the group structure in part (b) is functorial with respect to ring homomorphisms \( R \rightarrow R' \).

In the language of algebraic geometry, there is an affine scheme \( \text{Spec}(A) \) over \( \mathbb{Z} \) associated to \( A \). Any scheme \( X \) over a commutative ring \( k \) induces a functor \( X(\mathbb{Z}) \) from \( \mathbb{Z} \)-algebras \( R \) to sets, where \( X(R) \) is the set of \( k \)-scheme morphisms \( \text{Spec}(R) \rightarrow X \), also called the set of \( R \)-valued points of \( X \). Morphisms of affine \( k \)-schemes \( \text{Spec}(R) \rightarrow \text{Spec}(A) \) are in functorial one-to-one correspondence with \( k \)-algebra homomorphisms \( A \rightarrow R \). Note that every commutative ring is a \( \mathbb{Z} \)-algebra in a unique way.

What this problem shows is that for \( A \) as in part (a), \( \text{Spec}(A) \) is a group scheme \( \mathbb{G}_m \) over \( \mathbb{Z} \) whose functor of points send \( R \) to the additive group \( R^n \).

2. Work out the analog of Problem 1 for the Laurent polynomial ring \( A = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), with each variable grouplike, meaning that \( \Delta x_i = x_i \otimes x_i \), and with antipode \( S(x_i) = x_i^{-1} \) and co-unit \( \varepsilon(f) = f(1) \).

3. The axioms of a non-commutative Hopf algebra (over a commutative ground ring \( k \)) are essentially the same as in the commutative case, except that the antipode \( S \) is required to be an \( k \)-algebra antihomomorphism (i.e., it reverses multiplication). The coproduct \( \Delta: A \rightarrow A \otimes A \) and co-unit \( \varepsilon: A \rightarrow k \) are ordinary algebra homomorphisms.

(a) Write down the axioms explicitly, using Sweedler notation \( \Delta f = \sum f_{(1)} \otimes f_{(2)} \) for the coproduct, recalling that they should formally make \( A \) a group object in the opposite of the category of \( k \)-algebras, with \( \otimes \) playing the role of a product.

(b) Show that, even in the commutative case, the antipode is a homomorphism from the coproduct to the opposite coproduct.

(c) Show that the axioms are self-dual. Hence, in particular, if \( k \) is a field and \( A \) is a finite-dimensional Hopf algebra, then \( A^* \) has the structure of a Hopf algebra.

(d) Prove that if the antipode \( S \) in a Hopf algebra \( A \) is invertible (this holds automatically in some cases, including commutative and co-commutative Hopf algebras), then \( S^{-1} \) is the antipode for the same algebra with opposite coproduct, and also the antipode for the algebra with the same coproduct and the opposite product. In particular, if \( A \) is either commutative or co-commutative, then \( S^2 \) is the identity.

4. Show that the group algebra \( kG \) of a finite group \( G \) has a co-commutative Hopf algebra structure in which the elements of \( g \) are grouplike, and that this makes \( kG \) dual to the Hopf algebra of \( k \)-valued functions on \( G \). This works for any commutative ring \( k \).
5. If $A$ is an infinite-dimensional Hopf algebra over a field $k$, its dual space $A^*$ becomes an algebra, but not a co-algebra, because $\Delta^*: A^* \to (A \otimes A)^*$ need not map $A^*$ into $A^* \otimes A^*$, which is a proper subspace of $(A \otimes A)^*$.

Let $A^* \subseteq A^*$ be the preimage $(\Delta^*)^{-1}(A^* \otimes A^*)$. Show that $A^*$ is a subalgebra of $A^*$ and that the Hopf algebra structure on $A$ induces a dual Hopf algebra structure on $A^*$. This algebra is called the Hopf dual of $A$.

6. Show that the free abelian group generated by the elements $x^n/n!$ in $\mathbb{Q}[x]$ is a $\mathbb{Z}$-subalgebra $D \subseteq \mathbb{Q}[x]$, called the divided power algebra in one variable. Show that $D$ has a natural co-commutative Hopf algebra structure, dual to the Hopf algebra structure on $\mathbb{Z}[x]$ in which $x$ is primitive. Duality in this case means a perfect pairing $D \otimes_{\mathbb{Z}} \mathbb{Z}[x] \to \mathbb{Z}$ such that the coproduct, co-unit and antipode in each algebra are dual to the product, unit and antipode in the other.

7. Consider the case of the pair of dual Hopf algebras $kG$ and $\mathcal{O}(G)$ in Problem 4, where $\mathcal{O}(G)$ is the algebra of functions $G \to k$, when $k$ is a field of characteristic $p$ and $G = \mathbb{Z}/p\mathbb{Z}$ is a cyclic group of order $p$.

(a) Show that $k(G) = k[x]/(x^p - 1)$, with $x$ group-like. We can think of it as $\mathcal{O}(\mu_p)$ for the non-reduced group scheme of `$p$-th roots of unity' over $k$.

(b) Show that linear representations of $G$ over $k$, or $kG$ modules, are the same as $\mathcal{O}(G)$ comodules, and that except for the fact that $G$ is not connected group, it behaves just like a unipotent linear algebraic group, in the equivalent senses that (i) the only irreducible $kG$ module is the trivial representation; (ii) in every finite-dimensional representation, $G$ acts by upper unitriangular matrices in some basis; (iii) $G$ acts unipotently on $\mathcal{O}(G)$.

(c) Show that $\mathcal{O}(G)$ modules, or $kG$ comodules, are the same as $(\mathbb{Z}/p\mathbb{Z})$-graded vector spaces. In particular, $\mathcal{O}(G)$ has $p$ distinct non-isomorphic one-dimensional irreducible modules, and every module is a direct sum of these. In this sense the group scheme $\mu_p$ is `reductive' and its representation theory in any characteristic resembles the characteristic zero representation theory of a cyclic group of order $p$.

8. (a) Write out explicitly the axioms of a right coaction $W \to W \otimes A$, where $A$ is a Hopf algebra over $k$ and $W$ is a $k$ module, dual to the axioms of a group action. You can assume $k$ is a field if you like, but the axioms are the same over any commutative ring.

(b) Verify in detail that if $G \times V \to V$ is a linear algebraic action of an affine algebraic group on a finite dimensional vector space $V$ (considered as an algebraic variety), and $\rho: \mathcal{O}(V) \to \mathcal{O}(V) \otimes \mathcal{O}(G)$ is the corresponding homomorphism of algebras of functions, then $\rho$ maps $V^*$ into $V^* \otimes \mathcal{O}(G)$ and makes $V^*$ a $\mathcal{O}(G)$ comodule. Note that $V^*$ is the subspace of $\mathcal{O}(V)$ consisting of linear functions.

(c) Verify in detail that every finite-dimensional right $\mathcal{O}(G)$ comodule $W$ arises from a unique linear algebraic action of $G$ on $V = W^*$ as in part (b), and more explicitly, that the right action of $g \in G$ on $W$ dual to the left action on $V$ is given by composing $\rho: W \to W \otimes \mathcal{O}(G)$ with the evaluation map $ev_g: \mathcal{O}(G) \to k$.

9. (a) Show that if $G$ is an algebraic group, the grouplike elements of the Hopf algebra $\mathcal{O}(G)$ are the 1-dimensional characters of $G$, that is, the group homomorphisms
$G \to \mathbb{G}_m(K) = K^\times$, considered as functions on $G$.

(b) More generally, show that if $A$ is a Hopf algebra over a commutative ring $k$, then grouplike elements of $A$ correspond naturally to $A$ comodules isomorphic to $k$ as a $k$ module.

10. Let $A$ be a Hopf algebra over $k$. Show that a $k$-linear map $\lambda: A \to k$ is a grouplike element of the Hopf dual of $A$ if and only if is an algebra homomorphism.

In the case $A = \mathcal{O}(G)$ for an affine algebraic group $G$, this means that the grouplike elements of the Hopf dual of $A$ correspond to the group elements $g \in G$.

11. Let $G$ be an affine algebraic group and let $A = \mathcal{O}(G)$. Recall that $G$ embeds in the the algebra $A^*$ by $g \to \text{ev}_g$ and that its Lie algebra $\mathfrak{g} = T_eG$ embeds in $A^*$ as the space of linear functionals that kill the ideal $\mathfrak{m}_e^2$ and the constant functions. Verify in detail that the adjoint action of $G$ on $\mathfrak{g}$, given for $g \in G$ by the differential at $e$ of conjugation by $g$ on $G$, corresponds to conjugation by $G$ on $\mathfrak{g}$ in $A^*$.

12. Show that in characteristic zero, the finite dimensional representations of $GL_2(K)$ are completely reducible, and the irreducible representations are the standard representations on homogeneous polynomials of each degree $d$ in $K[x,y]$, tensored with integer powers of the 1-dimensional representation whose character is the determinant. More precisely, with $T \subset B \subset GL_2$ the diagonal and upper triangular matrices, the weight lattice is $X = \mathbb{Z}^2$, the dominant weights are $(\lambda_1, \lambda_2)$ such that $\lambda_1 \geq \lambda_2$, and $V_\lambda = (\det)^{\otimes \lambda_1} \otimes K[x,y]_{\lambda_1-\lambda_2}$ is the irreducible representation with highest weight $\lambda$.

More explicitly, show that the coordinate ring $\mathcal{O}(GL_2)$ decomposes into a direct sum of subspaces spanned by the matrix coefficients of the representations described above. This and the irreducibility of these representations implies the other conclusions.