# Math 261B Thurs. Nov. 12

$B \subset SL_2$ $\quad \mathcal{O}(B)$ vs. $\mathcal{U}(\mathfrak{b})$

$\mathcal{O}(B) = k[t^{\pm 1}, x]$ $\qquad \Delta t = t \otimes t$

$\qquad \Delta x = x \otimes t^{-1} + t \otimes x$

$B = \left\{ \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} \right\}$

$\mathfrak{b} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} = aH + bE \right\}$ $\qquad \mathcal{U}(\mathfrak{b}) = k\langle H, E\rangle / ([H,E] - 2E)$

$\Delta H = H \otimes 1 + 1 \otimes H$

$\Delta E = E \otimes 1 + 1 \otimes E$

Pairing $\quad \mathcal{U}(\mathfrak{b}) \otimes \mathcal{O}(B) \to k$

On basis $t^a x^m$ of $\mathcal{O}(B)$ $\qquad H : t^a \mapsto a \quad$ others $\mapsto 0$

$E : t^a x \mapsto 1 \quad$ others $\mapsto 0$

$t\frac{\partial}{\partial t}\Big|_{(1,0)}$ $\quad HE$, etc. determined $\Delta$ on $\mathcal{O}(B)$ $\qquad HE : t^a x \mapsto a+1 \quad$ (others $\to 0$)

$\quad \shortparallel$ $\qquad EH : t^a x \mapsto a-1 \quad$ "

$H = \frac{\partial}{\partial t}\Big|_{(1,0)} \quad E = \frac{\partial}{\partial x}\Big|_{(1,0)}$ $\qquad [H,E] : t^a x \mapsto 2 \quad$ (others $\to 0$)

$\qquad$ " $2E$

are point derivations at $I$ on $\mathcal{O}(B)$ $\Rightarrow \Delta H = H \otimes 1 + 1 \otimes H, \ldots$

"<u>Quantum Tori</u>" $\quad \mathcal{A} = k(q)$ (could be any comm ring $\mathcal{A}$ with $q^{\pm 1} \in \mathcal{A}$, e.g. $\mathbb{Z}[q^{\pm 1}]$)

$T$ w/ character lattice $X \cong \mathbb{Z}^n$,

define $\mathcal{O}_q(T) = \mathcal{A}\cdot X \cong \mathcal{A}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ $\quad$ basis $x^\lambda$ $\quad \lambda \in X$.

$$\Delta x^\lambda = x^\lambda \otimes x^\lambda$$

$T^\vee$ = torus w/ char lattice $X^\vee \cong \mathbb{Z}^n$

$$\mathcal{O}_q(T^\vee) = \mathcal{A}\cdot X^\vee = \mathcal{A}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \qquad z^\xi \quad \xi \in X^*$$

Pairing $\quad \mathcal{O}_q(T^\vee) \underset{\mathcal{A}}{\otimes} \mathcal{O}_q(T) \to \mathcal{A}$

$$(z^\xi, x^\lambda) = q^{\langle \xi, \lambda\rangle}$$

$$(z^{\xi_1} \otimes z^{\xi_2}, \Delta x^\lambda) = (z^{\xi_1} \otimes z^{\xi_2}, x^\lambda \otimes x^\lambda) = q^{\langle \xi_1, \lambda\rangle} q^{\langle \xi_2, \lambda\rangle}$$

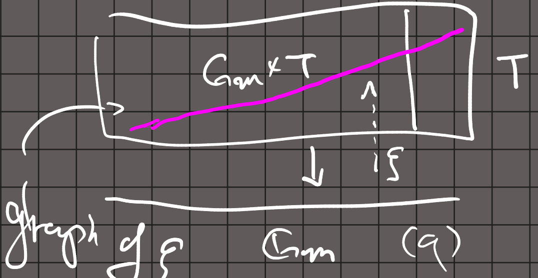
$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = q^{\langle \xi_1 + \xi_2, \lambda\rangle}$$

$$(z^{\xi_1} \cdot z^{\xi_2}, x^\lambda) = q^{\langle \xi_1 + \xi_2, \lambda\rangle}$$

It's a <u>Hopf pairing</u>

If $\mathcal{A} = k[q^{\pm 1}]$ then $\mathcal{O}_q(T) = k[q^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

$$= \mathcal{O}_k(\mathbb{G}_m \times T)$$




$\subset \mathbb{G}_m \times T$

cochar $\xi \in X^*$ is a homomorphism $\mathbb{G}_m \to T$

$\mathbb{G}_m \xrightarrow{\xi} T$

$\mathcal{O}_k(\mathbb{G}_m \times T) \to \mathcal{O}_k(\mathbb{G}_m)$

$q \mapsto q$

$x^\lambda \mapsto x^\lambda(\xi(q)) = q^{\langle \xi, \lambda \rangle}$

extend to

$\xi \in X^*$

$\downarrow T$

$\cup \quad \mathcal{O}_q(T)^*$

$\mathcal{O}_q(T) \to A$

$\uparrow \xi$ homomorphism

$X^* \to (\mathcal{O}_q(T)^*)^\times$

$A X^* \to \mathcal{O}_q(T)^*$

$\| $

$\mathcal{O}_q(T^\vee)$

$T \hookrightarrow \mathcal{O}(T)^*$

$g \mapsto e_g$

**Step 1** Replace $\mathcal{U}(\mathfrak{t})$ with $\mathcal{O}_q(T^\vee)$ in $\mathcal{U}(\mathfrak{b})$.

$A = k(q)$

New "$\mathcal{O}(B)$" $A[t^{\pm 1}, x]$

$\Delta t = t \otimes t$

$\Delta x = x \otimes t^{-1} + t \otimes x$

New "$\mathcal{U}(\mathfrak{b})$" $\subset$ "$\mathcal{O}(B)^*$" gen. by $E$ (as before)

and $K =$ "$q^H$"

evaluation on $\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$

$K : t^a \mapsto q^a$ others $\mapsto 0$

Old relation $[H, E] = 2E$ becomes $KEK^{-1} = q^2 E$

- General principle : $\exp u(\operatorname{ad} H) = \operatorname{Ad} \exp(uH)$ (Exercise)

$[H, -]$ $\qquad$ $\operatorname{Ad} K$ is $K(-)K^{-1}$

think of $u = \log q$ $\qquad$ $K = \exp(uH)$

$KEK^{-1} = q^2 E$

- Or check by hand using $\Delta$ in $\mathcal{O}(B)$
- $\mathcal{O}(B)$ comodules $V$ (hence $\mathcal{O}(T)$ comodules) have weight spaces

$V_m$ $\quad$ $H$ acts as $m$ ; $\quad$ $\underset{\parallel}{q^H}$ should act as $q^m$
$\phantom{q^H}K$

$V_m \xrightarrow[E]{} V_{m+2}$

$K E \underset{V_m}{\underset{\cap}{v}} = q^{m+2} E v = q^2 E K v$

$Kv = q^m v$

$KE = q^2 EK$

$KEK^{-1} = q^2 E$

$\mathcal{O}_q(T^\vee) \quad \mathcal{O}_q(T)$

$A[K^{\pm 1}] \qquad A[t^{\pm 1}] \overset{x \mapsto 0}{\twoheadleftarrow} \mathcal{O}(B)$

New "$U(\mathfrak{b})$" $\quad A\langle K^{\pm1}, E\rangle / (KEK^{-1} - q^2E)$ $\quad U(\mathfrak{h}) : H$

$\mathcal{O}(T^\vee) \quad \mathcal{O}(T)$ $\quad \Delta K = K\otimes K$ $\quad \mathcal{O}_q(T^\vee)^*$ $\quad K = \text{"}q^H\text{"}$

$A[k^{\pm1}] \quad A[t]$ $\quad \Delta E = E\otimes 1 + 1\otimes E$

Exercise

$(k^a, t^b) = q^{ab}$ $\quad$ "$\mathcal{O}(B)$" $\quad A[t^{\pm1}, x]$

$H$ primitive $\Leftrightarrow$ formally $\exp tH$ is group-like

$\Delta t = t\otimes t$

$\Delta x = x\otimes t^{-1} + t\otimes x$

Step 2 $\left(\begin{array}{l} q = v^2 \\ K = (K^{1/2})^2 \end{array}\right)$

Prop. There's a well-defined Hopf alg over $A = k(v)$ $\quad (q = v^2)$

$$U_v(\mathfrak{b}) = A\langle K^{\pm 1/2}, E\rangle / (K^{1/2} E K^{-1/2} - qE)$$

$$\Delta K^{1/2} = K^{1/2}\otimes K^{1/2}$$

$$\Delta E = E\otimes K^{-1/2} + K^{1/2}\otimes E$$

Proof Can check coassociativity on $K^{1/2}$, $E$

$$\Delta^{(3)} K^{1/2} = K^{1/2}\otimes K^{1/2}\otimes K^{1/2} \quad \ldots$$

$$\Delta^{(m)} E = \sum K^{1/2}\otimes\cdots\otimes K^{1/2}\otimes E\otimes K^{-1/2}\otimes K^{-1/2} \cdots \otimes K^{-1/2}$$

Check compatibility with relation:

$$(\Delta K^{1/2})(\Delta E)(\Delta K^{-1/2}) \overset{?}{=} q\,\Delta E$$

$$(K^{1/2} \otimes K^{1/2})(E \otimes K^{-1/2} + K^{1/2} \otimes E)(K^{-1/2} \otimes K^{-1/2})$$

$$= K^{1/2} E K^{-1/2} \otimes K^{-1/2} + K^{1/2} \otimes K^{1/2} E K^{-1/2}$$

$$= q\left(E \otimes K^{-1/2} + K^{1/2} \otimes E\right)$$

$q \mapsto 1$ : $\quad U_q(b) \approx \mathcal{O}(B)$

$$KE = q^2 EK$$

$$H_q = \frac{K - K^{-1}}{q - q^{-1}} \qquad [H_q, E] = \frac{(q^2-1)EK + (1-q^{-2})EK^{-1}}{q-q^{-1}}$$

$$KE - EK = (q^2-1)EK$$

$$\frac{q^2 - q^{-2}}{q - q^{-1}} E$$

$$\downarrow K \mapsto 1$$

$$(m)_q = \frac{q^m - q^{-m}}{q - q^{-1}}$$

$$\rightsquigarrow (q + q^{-1})E$$

$q \to 1 \quad (m)_q \to m \qquad (2)_q$

$$(H, E) = 2E$$

$$\mathcal{U}_v(b) = \mathbb{A}\langle K^{\pm 1/2}, E\rangle / (K^{1/2} E K^{-1/2} - qE)$$

$$\Delta K^{1/2} = K^{1/2} \otimes K^{1/2}$$

$$\Delta E = E \otimes K^{-1/2} + K^{1/2} \otimes E$$

$$\tilde{E} = K^{1/2} E$$

$$\Delta \tilde{E} = \tilde{E} \otimes 1 + K \otimes \tilde{E}$$

$$T \subset B$$

$$x \mapsto 0$$

$$t \mapsto t$$

$$\mathcal{O}(B) \to \mathcal{O}(T)$$

$$\mathcal{O}_v(B) = \mathbb{A}\langle t^{\pm 1}, x\rangle / (txt^{-1} - qx)$$

$$\Delta t = t \otimes t$$

$$\Delta x = x \otimes t^{-1} + t \otimes x$$

$$\mathcal{O}_q(B) \to \mathcal{O}_q(T)$$

$$t \mapsto t$$

$$x \mapsto 0$$

$$\mathcal{O}_q(T^\vee) \cap \mathcal{U}_q(b)$$

Still have Hopf pairing

$$\langle K^b, t^a\rangle = q^{ab} \qquad \langle (K^{1/2})^b, t^a\rangle = v^{ab}$$

$$\langle K^b, t^a x^m\rangle = 0 \qquad m > 0$$

$$\langle E, t^a x\rangle = 1 \qquad \langle E, \text{else}\rangle = 0$$

$$\langle K^b, t^a\rangle = q^{ab}$$

$$\langle K^b, x\rangle = 0 \qquad \langle E, t^a\rangle = 0$$

$$\langle E, x\rangle = 1$$

$\mathcal{O}_v(B) \cong \mathcal{U}_v(b)$ is a self-dual Hopf algebra.

Define $U_q(sl_2)$ over $A = k(q)$

generators $E, F, K^{\pm 1}$,

relations $KEK^{-1} = q^2 E$ $KFK^{-1} = q^{-2} F$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} = "H_q"$$

Coproduct

$$\Delta K = K \otimes K$$

$$\Delta E = E \otimes 1 + K \otimes E$$

$$\Delta F = F \otimes K^{-1} + 1 \otimes F$$

Next time: check compatibility