Math 261A: Lie Groups, Fall 2010
Problems, Set 4

1. (a) Describe the map $\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Lie}(GL(n,\mathbb{R})) = M_n(\mathbb{R}) \to \mathrm{Vect}(\mathbb{R}^n)$ given by the infinitesimal action of $GL_n(\mathbb{R})$.

(b) Show that $\mathfrak{so}(n,\mathbb{R})$ is equal to the subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ consisting of elements whose infinitesimal action is a vector field tangential to the unit sphere in $\mathbb{R}^n$.

2. Prove that if G is a connected Lie group, with $\mathrm{Lie}(G) = \mathfrak{g}$, then the connected Lie subgroup $Z \subseteq G$ whose Lie algebra is the center of $\mathfrak{g}$ is equal to the identity component of the center of G.

3. (a) Show that if H is a normal Lie subgroup of G, then $\mathrm{Lie}(H)$ is an ideal in $\mathrm{Lie}(G)$.

(b) Show that if G is connected, the converse also holds.

4. Show that the kernel of a Lie group homomorphism $G \to H$ is a closed subgroup of G whose Lie algebra is equal to the kernel of the induced map $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$.

5. Show that the intersection of two Lie subgroups H_1, H_2 of a Lie group G can be given a canonical structure of Lie subgroup so that its Lie algebra is $\mathrm{Lie}(H_1) \cap \mathrm{Lie}(H_2) \subseteq \mathrm{Lie}(G)$.

6. Classify the 3-dimensional Lie algebras $\mathfrak{g}$ over an algebraically closed field k of characteristic zero by showing that if $\mathfrak{g}$ is not a direct product of smaller Lie algebras, then either

(i) $\mathfrak{g} \cong \mathfrak{sl}(2,k)$,

(ii) $\mathfrak{g}$ is isomorphic to the nilpotent *Heisenberg Lie algebra* $\mathfrak{h}$ with basis X, Y, Z such that Z is central and $[X,Y] = Z$, or

(iii) $\mathfrak{g}$ is isomorphic to a solvable algebra $\mathfrak{s}$ which is the semidirect product of the abelian algebra k^2 by an invertible derivation. In particular $\mathfrak{s}$ has basis X, Y, Z such that $[Y,Z] = 0$, and $\mathrm{ad}\, X$ acts on $kY + kZ$ by an invertible matrix, which, up to change of basis in $kY + kZ$ and rescaling X, can be taken to be either $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, or $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right]$ for some nonzero $\lambda \in k$.

7. (a) Show that the Heisenberg Lie algebra $\mathfrak{h}$ in the preceding problem has the property that Z acts nilpotently in every finite-dimensional module, and as zero in every simple finite-dimensional module.

(b) Construct a simple infinite-dimensional $\mathfrak{h}$-module in which Z acts as a non-zero scalar. Hint: take X and Y to be the operators d/dt and t on $k[t]$.

8. Construct a simple 2-dimensional module for the Heisenberg algebra $\mathfrak{h}$ over any field k of characteristic 2. In particular, if $k = \overline{k}$, this gives a counterexample to Lie's theorem in non-zero characteristic.

9. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic zero, $\mathfrak{r}$ its radical.

(a) Prove that $[\mathfrak{g}, \mathfrak{r}]$ is contained in the nilradical (largest nilpotent ideal) of $\mathfrak{r}$.

(b) Deduce that the nilradical of $\mathfrak{g}$ is equal to the nilradical of $\mathfrak{r}$, and consists of all elements $x \in \mathfrak{r}$ such that $\mathrm{ad}\, x$ is nilpotent (on $\mathfrak{g}$ or equivalently on $\mathfrak{r}$).

10. (a) Show that $\mathrm{Ext}^1_{U(\mathfrak{g})}(k,k)$ can be canonically identified with the dual space of $\mathfrak{g}/\mathcal{D}\mathfrak{g}$.

(b) Part (a) implies that $(\mathfrak{g}/\mathcal{D}\mathfrak{g})^*$ is in canonical bijective correspondence with isomorphism classes of 2-dimensional $\mathfrak{g}$ modules V with a one-dimensional submodule W such that both W and V/W are trivial $\mathfrak{g}$ modules. Make this correspondence explicit.

11. Show that $\operatorname{Ext}^1_{U(\mathfrak{g})}(k, \mathfrak{g})$ can be canonically identified with the space $\operatorname{Der}(\mathfrak{g})/\operatorname{Inn}(\mathfrak{g})$ of derivations of $\mathfrak{g}$, modulo derivations of the form $\operatorname{ad} x$, and also with isomorphism classes of Lie algebras containing $\mathfrak{g}$ as a codimension 1 ideal.

12. Let $F(d)$ be the free Lie algebra on generators $X_1, \ldots, X_d$. It has a natural $\mathbb{N}^d$ grading in which $F(d)_{(k_1,\ldots,k_d)}$ is spanned by bracket monomials containing k_i occurences of each generator X_i. Use the PBW theorem to prove the generating function identity

$$\prod_{\mathbf{k}} \frac{1}{(1 - t_1^{k_1} \cdots t_d^{k_d})^{\dim F(d)_{(k_1,\ldots,k_d)}}} = \frac{1}{1 - (t_1 + \cdots + t_d)}.$$

13. Words in the symbols $X_1, \ldots, X_d$ form a monoid under concatentation, with identity the empty word. Define a *primitive word* to be a non-empty word that is not a power of a shorter word. A *primitive necklace* is an equivalence class of primitive words under rotation. Use the generating function identity in the preceding problem to prove that the dimension of $F(d)_{k_1,\ldots,k_d}$ is equal to the number of primitive necklaces in which each symbol X_i appears k_i times.

14. A *Lyndon word* is a primitive word that is the lexicographically least representative of its primitive necklace.

(a) Prove that w is a Lyndon word if and only if w is lexicographically less than v for every factorization $w = uv$ such that u and v are non-empty.

(b) Prove that if $w = uv$ is a Lyndon word of length > 1 and v is the longest proper right factor of w which is itself a Lyndon word, then u is also a Lyndon word. This factorization of w is called its *right standard factorization.*

(c) To each Lyndon word w in symbols $X_1, \ldots, X_d$ associate the bracket polynomial $p_w = X_i$ if $w = X_i$ has length 1, or, inductively, $p_w = [p_u, p_v]$, where $w = uv$ is the right standard factorization, if w has length > 1.

Prove that the elements p_w form a basis of $F(d)$.

15. Prove that if q is a power of a prime, then the dimension of the subspace of total degree $k_1 + \cdots + k_q = n$ in $F(q)$ is equal to the number of monic irreducible polynomials of degree n over the field with q elements.

16. Show that if $\mathfrak{g}$ acts by derivations on two algebras V and W, then its action on $V \otimes W$ is also by derivations (with the algebra structure on $V \otimes W$ such that $(v \otimes w)(v' \otimes w') = vv' \otimes ww'$).

17. (a) Show that every finite-dimensional $\mathfrak{g}$ module V has a unique maximal submodule which is a nil representation of $\mathfrak{g}$.

(b) Show that every finite-dimensional Lie algebra $\mathfrak{g}$ has a unique ideal $\mathfrak{j}$ such that $\mathfrak{j}$ is a nil representation of $\mathfrak{g}$, and the center of $\mathfrak{g}/\mathfrak{j}$ is trivial.

18. Given $k_1 + ... + k_r = n$, let $\mathfrak{h} \subseteq \mathfrak{gl}_n$ be the subalgebra consisting of block upper-triangular matrices with block sizes k_i. Describe the radical $\mathfrak{r}$ of $\mathfrak{h}$ and find a Levi decomposition $\mathfrak{h} = \mathfrak{s} \ltimes \mathfrak{r}$, where $\mathfrak{s}$ is a semisimple subalgebra. Take as known the fact that $\mathfrak{sl}_m$ is simple.

19. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic zero, $\mathfrak{r}$ its radical. Prove that $\mathcal{D}\mathfrak{g} \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}]$. (Hint: use Levi's theorem).