1. Let $SL(2, \mathbb{C})$ act on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations 
\[
\begin{bmatrix}
 a & b \\
 c & d \\
\end{bmatrix}
\begin{bmatrix}
 z \\
\end{bmatrix}
= 
\begin{bmatrix}
 az+b \\
 cz+d \\
\end{bmatrix}.
\]
Determine explicitly the vector fields $f(z)\partial_z$ corresponding to the infinitesimal action of the basis elements
\[
E = \begin{bmatrix}
 0 & 1 \\
 0 & 0 \\
\end{bmatrix}, \quad H = \begin{bmatrix}
 1 & 0 \\
 0 & -1 \\
\end{bmatrix}, \quad F = \begin{bmatrix}
 0 & 0 \\
 1 & 0 \\
\end{bmatrix},
\]
of $\mathfrak{sl}(2, \mathbb{C})$, and check that you have constructed a Lie algebra homomorphism by computing the commutators of these vector fields.

2. (a) Describe the map $\mathfrak{gl}(n, \mathbb{R}) = \text{Lie}(GL(n, \mathbb{R})) = M_n(\mathbb{R}) \to \text{Vect}(\mathbb{R}^n)$ given by the infinitesimal action of $GL_n(\mathbb{R})$.
(b) Show that $\mathfrak{so}(n, \mathbb{R})$ is equal to the subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ consisting of elements whose infinitesimal action is a vector field tangential to the unit sphere in $\mathbb{R}^n$.

3. (a) Let $X$ be an analytic vector field on $M$ all of whose integral curves are unbounded (i.e., they are defined on all of $\mathbb{R}$). Show that there exists an analytic action of $R = (\mathbb{R}, +)$ on $M$ such that $X$ is the infinitesimal action of the generator $\partial t$ of $\text{Lie}(R)$.
(b) More generally, prove the corresponding result for a family of $n$ commuting vector fields $X_i$ and action of $R^n$.

4. (a) Show that the matrix \[
\begin{bmatrix}
 -a & 0 \\
 0 & -b \\
\end{bmatrix}
\]
belongs to the identity component of $GL(2, \mathbb{R})$ for all positive real numbers $a$, $b$.
(b) Prove that if $a \neq b$, the above matrix is not in the image $\exp(\mathfrak{gl}(2, \mathbb{R}))$ of the exponential map.

5. (a) Show that the Lie algebra $\mathfrak{so}(3, \mathbb{C})$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.
(b) Construct a Lie group homomorphism $SL(2, \mathbb{C}) \to SO(3, \mathbb{C})$ which realizes the isomorphism of Lie algebras in part (a), and calculate its kernel.

6. (a) Show that the Lie algebra $\mathfrak{so}(4, \mathbb{C})$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$.
(b) Construct a Lie group homomorphism $SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \to SO(4, \mathbb{C})$ which realizes the isomorphism of Lie algebras in part (a), and calculate its kernel.

7. Show that the intersection of two Lie subgroups $H_1$, $H_2$ of a Lie group $G$ can be given a canonical structure of Lie subgroup so that its Lie algebra is $\text{Lie}(H_1) \cap \text{Lie}(H_2) \subseteq \text{Lie}(G)$.

8. Show that the kernel of a Lie group homomorphism $G \to H$ is a closed subgroup of $G$ whose Lie algebra is equal to the kernel of the induced map $\text{Lie}(G) \to \text{Lie}(H)$.

9. (a) Show that if $H$ is a normal Lie subgroup of $G$, then $\text{Lie}(H)$ is a Lie ideal in $\text{Lie}(G)$.
(b) Conversely, show that if $G$ is a connected Lie group, and $H$ a connected Lie subgroup, then $H$ is normal if $\text{Lie}(H)$ is an ideal.
10. Prove that every element of $SL_2(\mathbb{R})$ can be factored uniquely as a product of matrices of the form

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

where $z > 0$. Deduce that the map $\mathbb{R}^2 \times \mathbb{R}_+ \to SL_2(\mathbb{R})$ sending $(\theta, x, z)$ to the above element identifies $\mathbb{R}^2 \times \mathbb{R}_+$ with the universal covering space of $SL_2(\mathbb{R})$. Can you give the group law on the universal covering group explicitly in terms of the coordinates $(\theta, x, z)$?

[Later we’ll see that every finite-dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ comes from a representation of $SL_2(\mathbb{R})$, which implies that the covering group $\tilde{SL}_2(\mathbb{R})$ has no faithful linear representation, i.e., it is not a Lie subgroup of any $GL_n(\mathbb{R})$.]