Math 261A: Lie Groups, Fall 2010
Problems, Set 3

1. Let $SL(2,\mathbb{C})$ act on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = (az+b)/(cz+d)$. Determine explicitly the vector fields $f(z)\partial z$ corresponding to the infinitesimal action of the basis elements

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

of $\mathfrak{sl}(2,\mathbb{C})$, and check that you have constructed a Lie algebra homomorphism by computing the commutators of these vector fields.

2. (a) Describe the map $\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Lie}(GL(n,\mathbb{R})) = M_n(\mathbb{R}) \to \mathrm{Vect}(\mathbb{R}^n)$ given by the infinitesimal action of $GL_n(\mathbb{R})$.

(b) Show that $\mathfrak{so}(n,\mathbb{R})$ is equal to the subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ consisting of elements whose infinitesimal action is a vector field tangential to the unit sphere in $\mathbb{R}^n$.

3. (a) Let X be an analytic vector field on M all of whose integral curves are unbounded (*i.e.*, they are defined on all of $\mathbb{R}$). Show that there exists an analytic action of $R = (\mathbb{R}, +)$ on M such that X is the infinitesimal action of the generator ∂t of $\mathrm{Lie}(R)$.

(b) More generally, prove the corresponding result for a family of n commuting vector fields X_i and action of R^n.

4. (a) Show that the matrix $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$ belongs to the identity component of $GL(2,\mathbb{R})$ for all positive real numbers a, b.

(b) Prove that if $a \neq b$, the above matrix is not in the image $\exp(\mathfrak{gl}(2,\mathbb{R}))$ of the exponential map.

5. (a) Show that the Lie algebra $\mathfrak{so}(3,\mathbb{C})$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

(b) Construct a Lie group homomorphism $SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$ which realizes the isomorphism of Lie algebras in part (a), and calculate its kernel.

6. (a) Show that the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$.

(b) Construct a Lie group homomorphism $SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to SO(4,\mathbb{C})$ which realizes the isomorphism of Lie algebras in part (a), and calculate its kernel.

7. Show that the intersection of two Lie subgroups H_1, H_2 of a Lie group G can be given a canonical structure of Lie subgroup so that its Lie algebra is $\mathrm{Lie}(H_1) \cap \mathrm{Lie}(H_2) \subseteq \mathrm{Lie}(G)$.

8. Show that the kernel of a Lie group homomorphism $G \to H$ is a closed subgroup of G whose Lie algebra is equal to the kernel of the induced map $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$.

9. (a) Show that if H is a normal Lie subgroup of G, then $\mathrm{Lie}(H)$ is a Lie ideal in $\mathrm{Lie}(G)$.

(b) Conversely, show that if G is a connected Lie group, and H a connected Lie subgroup, then H is normal if $\mathrm{Lie}(H)$ is an ideal.

10. Prove that every element of $SL_2(\mathbb{R})$ can be factored uniquely as a product of matrices of the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

where $z > 0$. Deduce that the map $\mathbb{R}^2 \times \mathbb{R}_+ \to SL_2(\mathbb{R})$ sending (θ, x, z) to the above element identifies $\mathbb{R}^2 \times \mathbb{R}_+$ with the universal covering space of $SL_2(\mathbb{R})$. Can you give the group law on the universal covering group explicitly in terms of the coordinates (θ, x, z)?

[Later we'll see that every finite-dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ comes from a representation of $SL_2(\mathbb{R})$, which implies that the covering group $\widetilde{SL_2}(R)$ has no faithful linear representation, *i.e.*, it is not a Lie subgroup of any $GL_n(\mathbb{R})$.]