Math 261A: Lie Groups, Fall 2010
Problems, Set 2

1. Let G be the semidirect product $\mathbb{C} \ltimes \mathbb{C}$ where $(\mathbb{C}, +)$ acts on itself by $z \cdot w = e^z w$. Explicitly, $G = \mathbb{C}^2$ with the group law $(z, w)(z', w') = (z + z', w + e^z w')$.

(a) Identifying $\mathfrak{g} = \mathrm{Lie}(G) = T_1 G$ with $\mathbb{C}^2$ in the obvious way, let X and Y be the unit coordinate vectors. Show that the corresponding left invariant vector fields are $X = \partial/\partial z$ and $Y = e^z \partial/\partial w$, and that the Lie bracket is given by $[X, Y] = Y$.

(b) Calculate $\exp(aX + bY)$.

(c) Show the exponential map $\exp\colon \mathfrak{g} \to G$ is not surjective.

(d) Show that G has a closed 3-dimensional real Lie subgroup H isomorphic to $\mathbb{R}^3$ as a manifold, whose exponential map is also not surjective.

(e) Show that there is a homomorphism of Lie groups $H \to E(\mathbb{R}^2)$ with kernel a discrete subgroup isomorphic to $\mathbb{Z}$, and image the identity component $E_0(\mathbb{R}^2)$ consisting of orientation preserving Euclidean motions of $\mathbb{R}^2$. (In other words, H is the simply-connected covering group of $E_0(\mathbb{R}^2)$.) Is the exponential map of $E_0(\mathbb{R}^2)$ surjective?

2. Varadarajan, Ch. 2, Ex. 10

3. Varadarajan, Ch. 2, Ex. 18

4. Varadarajan, Ch. 2, Ex. 20

5. Varadarajan, Ch. 2, Ex. 39

6. Varadarajan, Ch. 2, Ex. 40

7. Using the fact that the differential $d\mu_{(1,1)}$ of multiplication in G is addition in $\mathfrak{g} = \mathrm{Lie}(G)$, show that if $\{x_1, \ldots, x_n\}$ is a basis of $\mathfrak{g}$, then the map $t_1 x_1 + \cdots + t_n x_n \mapsto \exp(t_1 x_1) \cdots \exp(t_n x_n)$ gives a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of 1 in G. (Coordinate charts around $1 \in G$ obtained in this way are called systems of canonical coordinates.)

8. (a) Prove that if G is a real Lie group, H is a closed subgroup, and we define $\mathcal{L}(H) = \{x \in \mathrm{Lie}(G) : \exp(tx) \in H \text{ for all } t \in \mathbb{R}\}$, then H is a regularly embedded submanifold, hence a Lie group, with $\mathrm{Lie}(H) = \mathcal{L}(H)$. Hint: imitate the proof we did in class for $G = GL_n$.

(b) If G is a complex Lie group, show that a closed subgroup H is a complex submanifold (and hence a complex Lie group) if and only if $\mathrm{Lie}(H)$ is a $\mathbb{C}$ vector subspace of $\mathrm{Lie}(G)$.

9. Prove that every discrete subgroup of a connected topological group (in particular, of a connected Lie group) is central.

10. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Prove that if $x, y \in G$ satisfy $[x, y] = 0$, then $\exp(x)\exp(y) = \exp(x + y)$.

11. Prove that a connected Lie group G is abelian if and only if its Lie algebra is abelian (this means that the Lie bracket is identically zero). Give an example to show that the hypothesis "connected" is needed.

12. Let G be a real Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. Suppose that $\mathfrak{g}$ is the underlying real vector space of a complex vector space $\mathfrak{g}_{\mathbb{C}}$ and that the Lie bracket is $\mathbb{C}$-linear each variable. Prove that G has a unique structure of complex Lie group such that $\mathfrak{g}_{\mathbb{C}}$ is its Lie algebra.