1. (a) Show that the stabilizer in $SL_n(\mathbb{C})$ of the first coordinate vector $e_1 \in \mathbb{C}^n$ is isomorphic to a semidirect product of $SL_{n-1}(\mathbb{C})$ with the additive group $\mathbb{C}^{n-1}$, and similarly over $\mathbb{R}$.

(b) Show that the orbit of $e_1$ is $\mathbb{C}^n \setminus \{0\}$ or $\mathbb{R}^n \setminus \{0\}$ (except for $n = 1$, when it’s a point).

(c) Deduce by induction on $n$ that $SL_n(\mathbb{C})$ and $SL_n(\mathbb{R})$ are connected.

2. (a) Show that $SO_n(\mathbb{R})$ and $SO_n(\mathbb{C})$ are connected.

(b) Show that $O_n(\mathbb{R})$ and $O_n(\mathbb{C})$ each have two connected components, the identity component being $SO_n$, and the other consisting of orthogonal matrices of determinant $-1$.

(c) Show that the center of $O_n$ is $\{\pm I_n\}$.

(d) Show that if $n$ is odd, then $SO_n$ has trivial center and $O_n \cong SO_n \times (\mathbb{Z}/2\mathbb{Z})$ as a Lie group.

(e) Show that if $n$ is even, then the center of $SO_n$ has two elements, and $O_n$ is a semidirect product $(\mathbb{Z}/2\mathbb{Z}) \rtimes SO_n$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $SO_n$ by a non-trivial outer automorphism of order 2.

3. Show that a closed subgroup $H \subseteq GL_n(\mathbb{C})$ is a regularly embedded $\mathbb{C}$ submanifold, and thus a complex Lie group, if and only if $\text{Lie}(H)$ is a $\mathbb{C}$ vector subspace of $\mathfrak{gl}_n(\mathbb{C})$.

4. (a) Let $\phi: S^2 \to \mathbb{CP}^1$ be the map given by stereographic projection from the north pole of $S^2$ to the complex plane $\mathbb{C}$, with $\phi$ mapping the south pole to 0, the equator to the unit circle $\{|z| = 1\}$, and the north pole to $\infty$. Verify that $\phi$ is an isometry between the standard angle metric on $S^2$ and the Fubini-Study metric on $\mathbb{CP}^1$ given by $d(x, y) = 2 \cos^{-1}|(x, y)|$, where $x, y \in \mathbb{C}^2$ are unit vectors.

(b) Work out the resulting Lie group homomorphism $\psi: U(2) \to SO(3)$ in explicit coordinates, i.e., find the entries of the $3 \times 3$ real orthogonal matrix $\psi(A)$ in terms of the entries of the $2 \times 2$ complex unitary matrix $A$.

5. Construct an isomorphism of $GL(n, \mathbb{C})$ (as a Lie group and an algebraic group) with a closed subgroup of $SL(n+1, \mathbb{C})$.

6. Show that the map $\mathbb{C}^* \times SL(n, \mathbb{C}) \to GL(n, \mathbb{C})$ given by $(z, g) \mapsto zg$ is a surjective homomorphism of Lie and algebraic groups, find its kernel, and describe the corresponding homomorphism of Lie algebras.

7. Let $N_n \subseteq GL(n, \mathbb{C})$ be the group of upper-triangular matrices with 1’s on the diagonal. Show that for this group, the exponential map is a diffeomorphism of the Lie algebra onto the group.

8. A real form of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_\mathbb{R}$ such that $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \oplus i\mathfrak{g}_\mathbb{R}$, or equivalently, such that the canonical map $\mathbb{C} \otimes_\mathbb{R} \mathfrak{g}_\mathbb{R} \to \mathfrak{g}$ given by scalar multiplication is an isomorphism. A real form of a (connected) complex closed linear group $G$ is a (connected) closed real subgroup $G_\mathbb{R}$ such that $\text{Lie}(G_\mathbb{R})$ is a real form of $\text{Lie}(G)$. 
(a) Show that $U(n)$ is a compact real form of $GL(n, \mathbb{C})$ and $SU(n)$ is a compact real form of $SL(n, \mathbb{C})$.

(b) Show that $SO(n)$ is a compact real form of $SO(n, \mathbb{C})$.

9. Show that if $H$ is a compact closed subgroup of $GL_n(\mathbb{C})$, then every $X \in \text{Lie}(H)$ has the property that $iX$ is diagonalizable (over $\mathbb{C}$) and has real eigenvalues. Assuming $H$ is connected, does the converse hold?

10. Let $X$ be a manifold and let $TX$ be the set of pairs $(P \in X, v \in T_P X)$. Let $\pi: TX \to X$ denote the projection on the first component. Show that $TX$ has a natural manifold structure such that $\pi$ is smooth, and the smooth sections $X \to TX$ of $\pi$ are the same thing as smooth vector fields on $X$. (The manifold $TX$ is called the tangent bundle of $X$.)

11. Verify the following in any category with products:

(a) An object with multiplication map $G \times G \to G$, unit pt $\to G$, and left and right inverses $i_L, i_R: G \to G$ is a group object, i.e., $i_L = i_R$.

(b) If $G$ and $H$ are group objects, then $G \times H$ is canonically a group object in such a way that the projections to $G$ and $H$ are group homomorphisms.

(c) A group object $G$ has a canonical opposite group structure $G^{op}$ such that the inverse map $i: G \to G^{op}$ is an isomorphism of group objects.