Math 261A: Lie Groups, Fall 2010
Problems, Set 1

1. (a) Show that the stabilizer in $SL_n(\mathbb{C})$ of the first coordinate vector $e_1 \in \mathbb{C}^n$ is isomorphic to a semidirect product of $SL_{n-1}(\mathbb{C})$ with the additive group $\mathbb{C}^{n-1}$, and similarly over $\mathbb{R}$.

(b) Show that the orbit of e_1 is $\mathbb{C}^n \setminus \{0\}$ or $\mathbb{R}^n \setminus \{0\}$ (except for $n = 1$, when it's a point).

(c) Deduce by induction on n that $SL_n(\mathbb{C})$ and $SL_n(\mathbb{R})$ are connected.

2. (a) Show that $SO_n(\mathbb{R})$ and $SO_n(\mathbb{C})$ are connected.

(b) Show that $O_n(\mathbb{R})$ and $O_n(\mathbb{C})$ each have two connected components, the identity component being SO_n, and the other consisting of orthogonal matrices of determinant -1.

(c) Show that the center of O_n is $\{\pm I_n\}$.

(d) Show that if n is odd, then SO_n has trivial center and $O_n \cong SO_n \times (\mathbb{Z}/2\mathbb{Z})$ as a Lie group.

(e) Show that if n is even, then the center of SO_n has two elements, and O_n is a semidirect product $(\mathbb{Z}/2\mathbb{Z}) \ltimes SO_n$, where $\mathbb{Z}/2\mathbb{Z}$ acts on SO_n by a non-trivial outer automorphism of order 2.

3. Show that a closed subgroup $H \subseteq GL_n(\mathbb{C})$ is a regularly embedded $\mathbb{C}$ submanifold, and thus a complex Lie group, if and only if $\mathrm{Lie}(H)$ is a $\mathbb{C}$ vector subspace of $\mathfrak{gl}_n(\mathbb{C})$.

4. (a) Let $\phi\colon S^2 \to \mathbb{CP}^1$ be the map given by stereographic projection from the north pole of S^2 to the complex plane $\mathbb{C}$, with ϕ mapping the south pole to 0, the equator to the unit circle $\{|z| = 1\}$, and the north pole to ∞. Verify that ϕ is an isometry between the standard angle metric on S^2 and the Fubini-Study metric on $\mathbb{CP}^1$ given by $d(\overline{x}, \overline{y}) = 2\cos^{-1} |(x, y)|$, where $x, y \in \mathbb{C}^2$ are unit vectors.

(b) Work out the resulting Lie group homomorphism $\psi\colon U(2) \to SO(3)$ in explicit coordinates, *i.e.*, find the entries of the 3×3 real orthogonal matrix $\psi(A)$ in terms of the entries of the 2×2 complex unitary matrix A.

5. Construct an isomorphism of $GL(n, \mathbb{C})$ (as a Lie group and an algebraic group) with a closed subgroup of $SL(n + 1, \mathbb{C})$.

6. Show that the map $\mathbb{C}^* \times SL(n, \mathbb{C}) \to GL(n, \mathbb{C})$ given by $(z, g) \mapsto zg$ is a surjective homomorphism of Lie and algebraic groups, find its kernel, and describe the corresponding homomorphism of Lie algebras.

7. Let $N_n \subseteq GL(n, \mathbb{C})$ be the group of upper-triangular matrices with 1's on the diagonal. Show that for this group, the exponential map is a diffeomorphism of the Lie algebra onto the group.

8. A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_{\mathbb{R}}$ such that that $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus i\mathfrak{g}_{\mathbb{R}}$, or equivalently, such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}$ given by scalar multiplication is an isomorphism. A *real form* of a (connected) complex closed linear group G is a (connected) closed real subgroup $G_{\mathbb{R}}$ such that $\mathrm{Lie}(G_{\mathbb{R}})$ is a real form of $\mathrm{Lie}(G)$.

(a) Show that $U(n)$ is a compact real form of $GL(n, \mathbb{C})$ and $SU(n)$ is a compact real form of $SL(n, \mathbb{C})$.

(b) Show that $SO(n)$ is a compact real form of $SO(n, \mathbb{C})$.

9. Show that if H is a compact closed subgroup of $GL_n(\mathbb{C})$, then every $X \in \mathrm{Lie}(H)$ has the property that iX is diagonalizable (over $\mathbb{C}$) and has real eigenvalues. Assuming H is connected, does the converse hold?

10. Let X be a manifold and let TX be the set of pairs $(P \in X, v \in T_P X)$. Let $\pi \colon TX \to X$ denote the projection on the first component. Show that TX has a natural manifold structure such that π is smooth, and the smooth sections $X \to TX$ of π are the same thing as smooth vector fields on X. (The manifold TX is called the *tangent bundle* of X.)

11. Verify the following in any category with products:

(a) An object with multiplication map $G \times G \to G$, unit pt $\to G$, and left and right inverses $i_L, i_R \colon G \to G$ is a group object, *i.e.*, $i_L = i_R$.

(b) If G and H are group objects, then $G \times H$ is canonically a group object in such a way that the projections to G and H are group homomorphisms.

(c) A group object G has a canonical opposite group structure G^{op} such that the inverse map $i \colon G \to G^{\mathrm{op}}$ is an isomorphism of group objects.