Math 261A: Lie Groups, Fall 2008
Problems, Set 6

1. Show that the simple complex Lie algebra $\mathfrak{g}$ with root system G_2 has a 7-dimensional matrix representation with the generators shown below.

$$e_1 = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \qquad f_1 = \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&2&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{bmatrix} \qquad f_2 = \begin{bmatrix} 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&2&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0 \end{bmatrix}$$

2. (a) Show that there is a unique Lie group G over $\mathbb{C}$ with Lie algebra of type G_2.

(b) Find explicit equations of G realized as the algebraic subgroup of $GL(7,\mathbb{C})$ whose Lie algebra is the image of the matrix representation in Problem 1.

3. Show that the simply connected complex Lie group with Lie algebra $\mathfrak{so}(2n,\mathbb{C})$ is a double cover $\mathrm{Spin}(2n,\mathbb{C})$ of $SO(2n,\mathbb{C})$, whose center Z has order four. Show that if n is odd, then Z is cyclic, and there are three connected Lie groups with this Lie algebra: $\mathrm{Spin}(2n,\mathbb{C})$, $SO(2n,\mathbb{C})$ and $SO(2n,\mathbb{C})/\{\pm I\}$. If n is even, then $Z \cong (\mathbb{Z}/2\mathbb{Z})^2$, and there are two more Lie groups with the same Lie algebra.

4. If G is an affine algebraic group, and $\mathfrak{g}$ its Lie algebra, show that the canonical algebra homormorphism $U(\mathfrak{g}) \to \mathcal{O}(G)^*$ identifies $U(\mathfrak{g})$ with the set of linear functionals on $\mathcal{O}(G)$ whose kernel contains a power of the maximal ideal $\mathfrak{m} = \ker(\mathrm{ev}_e)$.

5. Show that there is a unique Lie group over $\mathbb{C}$ with Lie algebra of type E_8. Find the dimension of its smallest matrix representation.

6. Construct a finite dimensional Lie algebra over $\mathbb{C}$ which is not the Lie algebra of any algebraic group over $\mathbb{C}$. [Hint: the adjoint representation of an algebraic group on its Lie algebra is algebraic.]