Math 261A: Lie Groups, Fall 2008
Problems, Set 4

1. Classify the 3-dimensional Lie algebras $\mathfrak{g}$ over an algebraically closed field k of characteristic zero by showing that if $\mathfrak{g}$ is not a direct product of smaller Lie algebras, then either

(i) $\mathfrak{g} \cong \mathfrak{sl}(2, k)$,

(ii) $\mathfrak{g}$ is isomorphic to the nilpotent *Heisenberg Lie algebra* $\mathfrak{h}$ with basis X, Y, Z such that Z is central and $[X, Y] = Z$, or

(iii) $\mathfrak{g}$ is isomorphic to a solvable algebra $\mathfrak{s}$ which is the semidirect product of the abelian algebra k^2 by an invertible derivation. In particular $\mathfrak{s}$ has basis X, Y, Z such that $[Y, Z] = 0$, and $\operatorname{ad} X$ acts on $kY + kZ$ by an invertible matrix, which, up to change of basis in $kY + kZ$ and rescaling X, can be taken to be either $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, or $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right]$ for some nonzero $\lambda \in k$.

1′. Following problems 28–35 in Knapp, Chapter I, classify the 3-dimensional Lie algebras over k when $\operatorname{char}(k) = 0$ but k is not necessarily algebraically closed.

2. (a) Show that the Heisenberg Lie algebra $\mathfrak{h}$ in Problem 1 has the property that Z acts nilpotently in every finite-dimensional module, and as zero in every simple finite-dimensional module.

(b) Construct a simple infinite-dimensional $\mathfrak{h}$-module in which Z acts as a non-zero scalar. [Hint: take X and Y to be the operators d/dt and t on $k[t]$.]

3. Construct a simple 2-dimensional module for the Heisenberg algebra $\mathfrak{h}$ over any field k of characteristic 2. In particular, if $k = \overline{k}$, this gives a counterexample to Lie's theorem in non-zero characteristic.

4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k.

(a) Show that the intersection $\mathfrak{n}$ of the kernels of all finite-dimensional simple $\mathfrak{g}$-modules can be characterized as the largest ideal of $\mathfrak{g}$ which acts nilpotently in every finite-dimensional $\mathfrak{g}$-module. It is called the *nilradical* of $\mathfrak{g}$.

(b) Show that the nilradical of $\mathfrak{g}$ is contained in $\mathfrak{g}' \cap \operatorname{rad}(\mathfrak{g})$.

(c) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subalgebra and V a $\mathfrak{g}$-module. Given a linear functional $\lambda \colon \mathfrak{h} \to k$, define the associated *weight space* to be $V_\lambda = \{v \in V : Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$. Assuming $\operatorname{char}(k) = 0$, adapt the proof of Lie's theorem to show that if $\mathfrak{h}$ is an ideal and V is finite-dimensional, then V_λ is a $\mathfrak{g}$-submodule of V.

(d) Show that if $\operatorname{char}(k) = 0$ then the nilradical of $\mathfrak{g}$ is equal to $\mathfrak{g}' \cap \operatorname{rad} \mathfrak{g}$. [Hint: assume without loss of generality that $k = \overline{k}$ and obtain from Lie's theorem that any finite-dimensional simple $\mathfrak{g}$-module V has a non-zero weight space for some weight λ on $\mathfrak{g}' \cap \operatorname{rad} \mathfrak{g}$. Then use (c) to deduce that $\lambda = 0$ if V is simple.]

5. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k, $\operatorname{char}(k) = 0$. Prove that the the largest nilpotent ideal of $\mathfrak{g}$ is equal to the set of elements of $\mathfrak{r} = \operatorname{rad} \mathfrak{g}$ which act nilpotently in the adjoint action on $\mathfrak{g}$, or equivalently on $\mathfrak{r}$. In particular, it is equal to the largest nilpotent ideal of $\mathfrak{r}$.

6. Prove that the Lie algebra $\mathfrak{sl}(2,k)$ of 2×2 matrices with trace zero is simple, over a field k of any characteristic $\neq 2$. In characteristic 2, show that it is nilpotent.

7. In this exercise, we'll deduce from the standard functorial properties of Ext groups and their associated long exact sequences that $\mathrm{Ext}^1(N,M)$ bijectively classifies extensions $0\to M\to V\to N\to 0$ up to isomorphism, for modules over any associative ring with unity.

(a) Let F be a free module with a surjective homomorphism onto N, so we have an exact sequence $0\to K\to F\to N\to 0$. Use the long exact sequence to produce an isomorphism of $\mathrm{Ext}^1(N,M)$ with the cokernel of $\mathrm{Hom}(F,M)\to\mathrm{Hom}(K,M)$.

(b) Given $\phi\in\mathrm{Hom}(K,M)$, construct V as the quotient of $F\oplus M$ by the graph of $-\phi$ (note that this graph is a submodule of $K\oplus M\subseteq F\oplus M$).

(c) Use the functoriality of Ext and the long exact sequences to show that the characteristic class in $\mathrm{Ext}^1(N,M)$ of the extension constructed in (b) is the element represented by the chosen ϕ, and conversely, that if ϕ represents the characteristic class of a given extension, then the extension constructed in (b) is isomorphic to the given one.

8. Calculate $\mathrm{Ext}^i(k,k)$ for all i for the trivial representation k of $\mathfrak{sl}(2,k)$, where $\mathrm{char}(k)=0$. Conclude that the theorem that $\mathrm{Ext}^i(M,N)=0$ for $i=1,2$ and all finite-dimensional representations M, N of a semi-simple Lie algebra $\mathfrak{g}$ does not extend to $i>2$.

9. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Show that $\mathrm{Ext}^1(k,k)$ can be canonically identified with the dual space of $\mathfrak{g}/\mathfrak{g}'$, and therefore also with the set of 1-dimensional $\mathfrak{g}$-modules, up to isomorphism.

10. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Show that $\mathrm{Ext}^1(k,\mathfrak{g})$ can be canonically identified with the quotient $\mathrm{Der}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$, where $\mathrm{Der}(\mathfrak{g})$ is the space of derivations of $\mathfrak{g}$, and $\mathrm{Inn}(\mathfrak{g})$ is the subspace of inner derivations, that is, those of the form $d(x)=[y,x]$ for some $y\in\mathfrak{g}$. Show that this also classifies Lie algebra extensions $\widehat{\mathfrak{g}}$ containing $\mathfrak{g}$ as an ideal of codimension 1.

11. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Show that there is a canonical isomorphism $\mathrm{Ext}^1(\mathfrak{g},k)\cong\mathrm{Ext}^2(k,k)\oplus S^2((\mathfrak{g}/\mathfrak{g}')^*)$ where S^2 denotes the second symmetric power. The first term classifies those $\mathfrak{g}$-module extentions $0\to k\to\widehat{\mathfrak{g}}\to\mathfrak{g}\to 0$ that are (one-dimensional, central) Lie algebra extensions.

Addendum: This problem turned out to be harder than I thought, and I'm not even sure that it's true.

Let's assume the ground field has $\mathrm{char}(k)\neq 2$, so we can distinguish between symmetric and antisymmetric forms.

The weaker result that there is a canonical injection $\mathrm{Ext}^2(k,k)\oplus S^2((\mathfrak{g}/\mathfrak{g}')^*)\hookrightarrow\mathrm{Ext}^1(\mathfrak{g},k)$ can be proven by representing a 1-cocyle as a bilinear form on $\mathfrak{g}$ and considering the cases where the form is antisymmetric or symmetric.

For the stronger result, note that the identity $([x,z],z)=0$ holds for the symmetrization of the form representing a 1-cocycle. Then $([x,y],z)+(y,[x,z]])=([x,y+z],y+z)-([x,y],y)-([x,z],z)=0$, so the symmetrized form is invariant. Among the invariant symmetric forms are those whose radical contains $\mathfrak{g}'$. These represent 1-cocycles. But some further argument is needed to show that no other invariant form can arise by symmetrizing a 1-cocyle.

12. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k, $\mathrm{char}(k) = 0$. The *Malcev-Harish-Chandra* theorem says that all Levi subalgebras $\mathfrak{s} \subseteq \mathfrak{g}$ are conjugate under the action of the group $\exp \operatorname{ad} \mathfrak{n}$, where $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{g}$ (note that $\mathfrak{n}$ acts nilpotently on $\mathfrak{g}$, so the power series expression for $\exp \operatorname{ad} X$ reduces to a finite sum when $X \in \mathfrak{n}$).

(a) Show that the reduction we used to prove Levi's theorem by induction in the case where the radical $\mathfrak{r} = \operatorname{rad} \mathfrak{g}$ is not a minimal ideal also works for the Malcev-Harish-Chandra theorem. More precisely, show that if $\mathfrak{r}$ is nilpotent, the reduction can be done using any nonzero ideal $\mathfrak{m}$ properly contained in $\mathfrak{r}$. If $\mathfrak{r}$ is not nilpotent, use Problem 4 to show that $[\mathfrak{g}, \mathfrak{r}] \neq \mathfrak{r}$, then make the reduction by taking $\mathfrak{m}$ to contain $[\mathfrak{g}, \mathfrak{r}]$.

(b) In general, given a semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is an abelian ideal, show that $\mathrm{Ext}^1_{U(\mathfrak{h})}(k, \mathfrak{m})$ classifies subalgebras complementary to $\mathfrak{m}$, up to conjugacy by the action of $\exp \operatorname{ad} \mathfrak{m}$. Then use the vanishing of $\mathrm{Ext}^1(M, N)$ for finite-dimensional modules over a semi-simple Lie algebra to complete the proof of the Malcev-Harish-Chandra theorem.