1. Classify the 3-dimensional Lie algebras $\mathfrak{g}$ over an algebraically closed field $k$ of characteristic zero by showing that if $\mathfrak{g}$ is not a direct product of smaller Lie algebras, then either
   (i) $\mathfrak{g} \cong \mathfrak{sl}(2, k)$,
   (ii) $\mathfrak{g}$ is isomorphic to the nilpotent Heisenberg Lie algebra $\mathfrak{h}$ with basis $X, Y, Z$ such that $Z$ is central and $[X, Y] = Z$, or
   (iii) $\mathfrak{g}$ is isomorphic to a solvable algebra $\mathfrak{s}$ which is the semidirect product of the abelian algebra $k^2$ by an invertible derivation. In particular $\mathfrak{s}$ has basis $X, Y, Z$ such that $[Y, Z] = 0$, and $\text{ad} X$ acts on $kY + kZ$ by an invertible matrix, which, up to change of basis in $kY + kZ$ and rescaling $X$, can be taken to be either $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, or $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for some nonzero $\lambda \in k$.

1'. Following problems 28–35 in Knapp, Chapter I, classify the 3-dimensional Lie algebras over $k$ when $\text{char}(k) = 0$ but $k$ is not necessarily algebraically closed.

2. (a) Show that the Heisenberg Lie algebra $\mathfrak{h}$ in Problem 1 has the property that $Z$ acts nilpotently in every finite-dimensional module, and as zero in every simple finite-dimensional module.
   (b) Construct a simple infinite-dimensional $\mathfrak{h}$-module in which $Z$ acts as a non-zero scalar. [Hint: take $X$ and $Y$ to be the operators $d/dt$ and $t$ on $k[t]$.]

3. Construct a simple 2-dimensional module for the Heisenberg algebra $\mathfrak{h}$ over any field $k$ of characteristic 2. In particular, if $k = \overline{k}$, this gives a counterexample to Lie’s theorem in non-zero characteristic.

4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $k$.
   (a) Show that the intersection $\mathfrak{n}$ of the kernels of all finite-dimensional simple $\mathfrak{g}$-modules can be characterized as the largest ideal of $\mathfrak{g}$ which acts nilpotently in every finite-dimensional $\mathfrak{g}$-module. It is called the nilradical of $\mathfrak{g}$.
   (b) Show that the nilradical of $\mathfrak{g}$ is contained in $\mathfrak{g}' \cap \text{rad}(\mathfrak{g})$.
   (c) Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subalgebra and $V$ a $\mathfrak{g}$-module. Given a linear functional $\lambda: \mathfrak{h} \to k$, define the associated weight space to be $V_\lambda = \{v \in V : Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$. Assuming $\text{char}(k) = 0$, adapt the proof of Lie’s theorem to show that if $\mathfrak{h}$ is an ideal and $V$ is finite-dimensional, then $V_\lambda$ is a $\mathfrak{g}$-submodule of $V$.
   (d) Show that if $\text{char}(k) = 0$ then the nilradical of $\mathfrak{g}$ is equal to $\mathfrak{g}' \cap \text{rad}(\mathfrak{g})$. [Hint: assume without loss of generality that $k = \overline{k}$ and obtain from Lie’s theorem that any finite-dimensional simple $\mathfrak{g}$-module $V$ has a non-zero weight space for some weight $\lambda$ on $\mathfrak{g}' \cap \text{rad}(\mathfrak{g})$. Then use (c) to deduce that $\lambda = 0$ if $V$ is simple.]

5. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $k$, $\text{char}(k) = 0$. Prove that the the largest nilpotent ideal of $\mathfrak{g}$ is equal to the set of elements of $\mathfrak{r} = \text{rad}(\mathfrak{g})$ which act nilpotently in the adjoint action on $\mathfrak{g}$, or equivalently on $\mathfrak{r}$. In particular, it is equal to the largest nilpotent ideal of $\mathfrak{r}$. 

6. Prove that the Lie algebra $sl(2, k)$ of $2 \times 2$ matrices with trace zero is simple, over a field $k$ of any characteristic $\neq 2$. In characteristic 2, show that it is nilpotent.

7. In this exercise, we’ll deduce from the standard functorial properties of Ext groups and their associated long exact sequences that Ext$^1(N, M)$ bijectively classifies extensions $0 \to M \to V \to N \to 0$ up to isomorphism, for modules over any associative ring with unity.

(a) Let $F$ be a free module with a surjective homomorphism onto $N$, so we have an exact sequence $0 \to K \to F \to N \to 0$. Use the long exact sequence to produce an isomorphism of Ext$^1(N, M)$ with the cokernel of Hom$(F, M) \to$ Hom$(K, M)$.

(b) Given $\phi \in$ Hom$(K, M)$, construct $V$ as the quotient of $F \oplus M$ by the graph of $-\phi$ (note that this graph is a submodule of $K \oplus M \subseteq F \oplus M$).

(c) Use the functoriality of Ext and the long exact sequences to show that the characteristic class in Ext$^1(N, M)$ of the extension constructed in (b) is the element represented by the chosen $\phi$, and conversely, that if $\phi$ represents the characteristic class of a given extension, then the extension constructed in (b) is isomorphic to the given one.

8. Calculate Ext$^i(k, k)$ for all $i$ for the trivial representation $k$ of $sl(2, k)$, where char($k$) = 0. Conclude that the theorem that Ext$^i(M, N) = 0$ for $i = 1, 2$ and all finite-dimensional representations $M, N$ of a semi-simple Lie algebra $g$ does not extend to $i > 2$.

9. Let $g$ be a finite-dimensional Lie algebra. Show that Ext$^1(k, k)$ can be canonically identified with the dual space of $g/g'$, and therefore also with the set of 1-dimensional $g$-modules, up to isomorphism.

10. Let $g$ be a finite-dimensional Lie algebra. Show that Ext$^1(k, g)$ can be canonically identified with the quotient Der$(g)/$Inn$(g)$, where Der$(g)$ is the space of derivations of $g$, and Inn$(g)$ is the subspace of inner derivations, that is, those of the form $d(x) = [y, x]$ for some $y \in g$. Show that this also classifies Lie algebra extensions $\hat{g}$ containing $g$ as an ideal of codimension 1.

11. Let $g$ be a finite-dimensional Lie algebra. Show that there is a canonical isomorphism Ext$^1(g, k) \cong$ Ext$^2(k, k) \oplus S^2((g/g')^*)$ where $S^2$ denotes the second symmetric power. The first term classifies those $g$-module extensions $0 \to k \to \hat{g} \to g \to 0$ that are (one-dimensional, central) Lie algebra extensions.

Addendum: This problem turned out to be harder than I thought, and I’m not even sure that it’s true.

Let’s assume the ground field has char($k$) $\neq 2$, so we can distinguish between symmetric and antisymmetric forms.

The weaker result that there is a canonical injection Ext$^2(k, k) \oplus S^2((g/g')^*) \hookrightarrow$ Ext$^1(g, k)$ can be proven by representing a 1-cocycle as a bilinear form on $g$ and considering the cases where the form is antisymmetric or symmetric.

For the stronger result, note that the identity $([x, z], z) = 0$ holds for the symmetrization of the form representing a 1-cocycle. Then $((x, y), z) + (y, [x, z]) = ([x, y + z], y + z) - (x, y) - ([x, z], z) = 0$, so the symmetrized form is invariant. Among the invariant symmetric forms are those whose radical contains $g'$. These represent 1-cocycles. But some further argument is needed to show that no other invariant form can arise by symmetrizing a 1-cocycle.
12. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $k$, $\text{char}(k) = 0$. The Malcev-Harish-Chandra theorem says that all Levi subalgebras $\mathfrak{s} \subseteq \mathfrak{g}$ are conjugate under the action of the group $\exp \text{ ad} \mathfrak{n}$, where $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{g}$ (note that $\mathfrak{n}$ acts nilpotently on $\mathfrak{g}$, so the power series expression for $\exp \text{ ad} X$ reduces to a finite sum when $X \in \mathfrak{n}$).

(a) Show that the reduction we used to prove Levi's theorem by induction in the case where the radical $\mathfrak{r} = \text{rad} \mathfrak{g}$ is not a minimal ideal also works for the Malcev-Harish-Chandra theorem. More precisely, show that if $\mathfrak{r}$ is nilpotent, the reduction can be done using any nonzero ideal $\mathfrak{m}$ properly contained in $\mathfrak{r}$. If $\mathfrak{r}$ is not nilpotent, use Problem 4 to show that $[\mathfrak{g}, \mathfrak{r}] \neq \mathfrak{r}$, then make the reduction by taking $\mathfrak{m}$ to contain $[\mathfrak{g}, \mathfrak{r}]$.

(b) In general, given a semidirect product $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{m}$, where $\mathfrak{m}$ is an abelian ideal, show that $\text{Ext}^1_{U(\mathfrak{h})}(k, \mathfrak{m})$ classifies subalgebras complementary to $\mathfrak{m}$, up to conjugacy by the action of $\exp \text{ ad} \mathfrak{m}$. Then use the vanishing of $\text{Ext}^1(M, N)$ for finite-dimensional modules over a semi-simple Lie algebra to complete the proof of the Malcev-Harish-Chandra theorem.