Math 261A: Lie Groups, Fall 2008
Problems, Set 2

1. (a) Show that the composition of two immersions is an immersion.

(b) Show that an immersed submanifold $N \subseteq M$ is always a closed submanifold of an open submanifold, but not necessarily an open submanifold of a closed submanifold.

2. Prove that if $f\colon N \to M$ is a smooth [analytic, holomorphic] map, then $(df)_p$ is surjective if and only if there are open neighborhoods U of p and V of $f(p)$, and an isomorphism $\psi\colon V \times W \to U$, such that $f \circ \psi$ is the projection on V.

In particular, deduce that the fibers of f meet a neighborhood of p in immersed closed submanifolds of that neighborhood.

3. [corrected 9/30] Prove the *implicit function theorem*: a map (of sets) $f : M \to N$ between manifolds is smooth [analytic, holomorphic] if and only if its graph H is an immersed closed submanifold of $M \times N$, and the tangent space to H at each point $(x, f(x))$ projects isomorphically on the tangent space $T_x M$.

4. Prove that the curve $y^2 = x^3$ in $\mathbb{R}^2$ is not an immersed submanifold. [This is a stronger statement than the observation we made in class that the smooth bijection $t \mapsto (t^2, t^3)$ of $\mathbb{R}$ onto this curve is not an immersion.]

5. Let M be a complex holomorphic manifold, p a point of M, X a holomorphic vector field. Show that X has a complex integral curve γ defined on an open neighborhood U of 0 in $\mathbb{C}$, and unique on U if U is connected, which satisfies the usual defining equation but in a complex instead of a real variable t.

Show that the restriction of γ to $U \cap \mathbb{R}$ is a real integral curve of X, when M is regarded as a real analytic manifold. [This exercise is meant to clarify a point left vague in the lecture.]

6. Let $SL(2, \mathbb{C})$ act on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = (az + b)/(cz + d)$. Determine explicitly the vector fields $f(z)\partial z$ corresponding to the infinitesimal action of the basis elements

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

of $\mathfrak{sl}(2, \mathbb{C})$, and check that you have constructed a Lie algebra homomorphism by computing the commutators of these vector fields.

7. (a) Describe the map $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{Lie}(GL(n, \mathbb{R})) = M_n(\mathbb{R}) \to \mathrm{Vect}(\mathbb{R}^n)$ given by the infinitesimal action of $GL_n(\mathbb{R})$.

(b) Show that $\mathfrak{so}(n, \mathbb{R})$ is equal to the subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ consisting of elements whose infinitesimal action is a vector field tangential to the unit sphere in $\mathbb{R}^n$.

8. (a) Let X be an analytic vector field on M all of whose integral curves are unbounded (*i.e.*, they are defined on all of $\mathbb{R}$). Show that there exists an analytic action of $R = (\mathbb{R}, +)$ on M such that X is the infinitesimal action of the generator ∂t of $\mathrm{Lie}(R)$.

(b) More generally, prove the corresponding result for a family of n commuting vector fields X_i and action of R^n.

9. (a) Show that the matrix $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$ belongs to the identity component of $GL(2, \mathbb{R})$ for all positive real numbers a, b.

(b) Prove that if $a \neq b$, the above matrix is not in the image $\exp(\mathfrak{gl}(2, \mathbb{R}))$ of the exponential map.