Math 261A: Lie Groups, Fall 2008
Problems, Set 1

1. (a) Show that the orthogonal groups $O_n(\mathbb{R})$ and $O_n(\mathbb{C})$ have two connected components, the identity component being the special orthogonal group SO_n, and the other consisting of orthogonal matrices of determinant -1.

(b) Show that the center of O_n is $\{\pm I_n\}$.

(c) Show that if n is odd, then SO_n has trivial center and $O_n \cong SO_n \times (\mathbb{Z}/2\mathbb{Z})$ as a Lie group.

(d) Show that if n is even, then the center of SO_n has two elements, and O_n is a semidirect product $(\mathbb{Z}/2\mathbb{Z}) \ltimes SO_n$, where $\mathbb{Z}/2\mathbb{Z}$ acts on SO_n by a non-trivial outer automorphism of order 2.

2. Problems 5-9 in Knapp Intro §6, which lead you through the construction of a smooth group homomorphism $\Phi\colon SU(2) \to SO(3)$ which induces an isomorphism of Lie algebras and identifies $SO(3)$ with the quotient of $SU(2)$ by its center $\{\pm I\}$.

3. Construct an isomorphism of $GL(n, \mathbb{C})$ (as a Lie group and an algebraic group) with a closed subgroup of $SL(n+1, \mathbb{C})$.

4. Show that the map $\mathbb{C}^* \times SL(n, \mathbb{C}) \to GL(n, \mathbb{C})$ given by $(z, g) \mapsto zg$ is a surjective homomorphism of Lie and algebraic groups, find its kernel, and describe the corresponding homomorphism of Lie algebras.

5. Find the Lie algebra of the group $U \subseteq GL(n, \mathbb{C})$ of upper-triangular matrices with 1 on the diagonal. Show that for this group, the exponential map is a diffeomorphism of the Lie algebra onto the group.

6. A *real form* of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_\mathbb{R}$ such that that $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \oplus i\mathfrak{g}_\mathbb{R}$, or equivalently, such that the canonical map $\mathfrak{g}_R \otimes_\mathbb{R} \mathbb{C} \to \mathfrak{g}$ given by scalar multiplication is an isomorphism. A *real form* of a (connected) complex closed linear group G is a (connected) closed real subgroup $G_\mathbb{R}$ such that $\mathrm{Lie}(G_\mathbb{R})$ is a real form of $\mathrm{Lie}(G)$.

(a) Show that $U(n)$ is a compact real form of $GL(n, \mathbb{C})$ and $SU(n)$ is a compact real form of $SL(n, \mathbb{C})$.

(b) Show that $SO(n)$ is a compact real form of $SO(n, \mathbb{C})$.

(c) Show that $Sp(n)$ is a compact real form of $Sp(n, \mathbb{C})$.

7. [corrected 9/30] Show that if H is a compact closed linear group, then every $X \in \mathrm{Lie}(H)$ has the property that iX is diagonalizable (over $\mathbb{C}$) and has real eigenvalues. Assuming H is connected, does the converse hold?