1. (a) Show that the orthogonal groups $O_n(\mathbb{R})$ and $O_n(\mathbb{C})$ have two connected components, the identity component being the special orthogonal group $SO_n$, and the other consisting of orthogonal matrices of determinant $-1$.

(b) Show that the center of $O_n$ is $\{\pm I_n\}$.

(c) Show that if $n$ is odd, then $SO_n$ has trivial center and $O_n \cong SO_n \times (\mathbb{Z}/2\mathbb{Z})$ as a Lie group.

(d) Show that if $n$ is even, then the center of $SO_n$ has two elements, and $O_n \cong SO_n \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $SO_n$ by a non-trivial outer automorphism of order 2.

2. Problems 5-9 in Knapp Intro §6, which lead you through the construction of a smooth group homomorphism $\Phi: SU(2) \rightarrow SO(3)$ which induces an isomorphism of Lie algebras and identifies $SO(3)$ with the quotient of $SU(2)$ by its center $\{\pm I\}$.

3. Construct an isomorphism of $GL(n, \mathbb{C})$ (as a Lie group and an algebraic group) with a closed subgroup of $SL(n+1, \mathbb{C})$.

4. Show that the map $\mathbb{C}^* \times SL(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$ given by $(z, g) \mapsto zg$ is a surjective homomorphism of Lie and algebraic groups, find its kernel, and describe the corresponding homomorphism of Lie algebras.

5. Find the Lie algebra of the group $U \subseteq GL(n, \mathbb{C})$ of upper-triangular matrices with 1 on the diagonal. Show that for this group, the exponential map is a diffeomorphism of the Lie algebra onto the group.

6. A real form of a complex Lie algebra $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_R$ such that $\mathfrak{g} = \mathfrak{g}_R \oplus i\mathfrak{g}_R$, or equivalently, such that the canonical map $\mathfrak{g}_R \otimes_\mathbb{R} \mathbb{C} \rightarrow \mathfrak{g}$ given by scalar multiplication is an isomorphism. A real form of a (connected) complex closed linear group $G$ is a (connected) closed real subgroup $G_\mathbb{R}$ such that $\text{Lie}(G_\mathbb{R})$ is a real form of $\text{Lie}(G)$.

(a) Show that $U(n)$ is a compact real form of $GL(n, \mathbb{C})$ and $SU(n)$ is a compact real form of $SL(n, \mathbb{C})$.

(b) Show that $SO(n)$ is a compact real form of $SO(n, \mathbb{C})$.

(c) Show that $Sp(n)$ is a compact real form of $Sp(n, \mathbb{C})$.

7. [corrected 9/30] Show that if $H$ is a compact closed linear group, then every $X \in \text{Lie}(H)$ has the property that $iX$ is diagonalizable (over $\mathbb{C}$) and has real eigenvalues. Assuming $H$ is connected, does the converse hold?