**Math 261A: Lie Groups, Fall 2015**
**Problem Set 2**

Problems from Varadarajan (assume characteristic zero as needed):

Chapter 3, Exercises 2, 3, 14, 21 ($\mathfrak{g}$ solvable should suffice), 26 (*reductive* means a product of a semisimple and an abelian Lie algebra), 27, 28, 29, 36.

Hint on 36: First show that the connected Lie subgroup $H$ corresponding to the center of $\mathrm{Lie}(G)$ is closed, simply connected and central, hence normal. Then use induction on $\dim(G)$ to conclude that any compact $K \subseteq G$ is contained in $H$, thus reducing the problem to the case that $G$ is abelian.

Other problems:

1. Let $U = \mathfrak{A}(\mathfrak{g})$ be the enveloping algebra of a Lie algebra $\mathfrak{g}$. Recall that, by the Poincaré-Birkhoff-Witt theorem, the graded algebra $\mathrm{gr}(U)$ associated to the canonical filtration $U_{\leq d} = \sum_{n \leq d} g^n$ is isomomorphic to the symmetric algebra $S(\mathfrak{g})$.

Assume now that $\mathfrak{g}$ is a Lie algebra over a field $k$ of characterstic zero. The *symmetrizer map* (Varadarajan, Sec. 3.3, p. 180) is then a linear isomorphism (but not an algebra homomorphism) $\lambda \colon S(\mathfrak{g}) \to U$.

(a) Let $\mathfrak{g}$ act on $S(\mathfrak{g})$ by the unique derivations extending the adjoint action of $\mathfrak{g}$ on itself, and on $U$ by commutators: $X \cdot u = Xu - uX$ for $X \in \mathfrak{g}$ and $u \in U$. Show that $\lambda$ is $\mathfrak{g}$-invariant.

(b) Let $Z = Z(U)$ be the center of $U$, and note that $Z = U^{\mathfrak{g}}$. Show that $\mathrm{gr}(Z) \subseteq \mathrm{gr}(U) = S(\mathfrak{g})$ is equal to $S(\mathfrak{g})^{\mathfrak{g}}$, where we filter $Z$ by $Z_{\leq d} = Z \cap U_{\leq d}$. In particular, by this and (a), $\lambda$ gives a linear isomorphism of $\mathrm{gr}(Z)$ onto $Z$.

(c) Either prove or make sure you understand the proof in the book of Varadarajan, Theorem 3.3.8 (b), which says that if homogeneous elements $T_1, \ldots, T_n$ generate $\mathrm{gr}(Z)$, then $\lambda(T_1), \ldots, \lambda(T_n)$ generate $Z$, and consequently if $\mathrm{gr}(Z)$ is isomorphic to a polynomial ring, then so is $Z$ (even though $\lambda$ need not be an algebra homomorphism).

2. (a) Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The adjoint action is an action of $G$ by Lie algebra automorphisms of $\mathfrak{g}$, hence induces an action of $G$ by algebra automorphisms on the enveloping algebra $\mathfrak{A}(\mathfrak{g})$. Prove that the subalgebra $\mathfrak{A}(\mathfrak{g})^G$ of $G$-invariant elements is contained in the center $Z(\mathfrak{A}(\mathfrak{g}))$, and that if $G$ is connected, then $\mathfrak{A}(\mathfrak{g})^G = Z(\mathfrak{A}(\mathfrak{g}))$.

(b) Show that $\mathfrak{A}(\mathfrak{g})^G$ can be identified with the algebra of differential operators on $G$ which are invariant under both the left and right actions of $G$ on itself.

(c) In the case $G = GL_n(\mathbb{C})$, use the fact that $\mathrm{tr}(XY)$ gives a non-degenerate invariant pairing on matrices to show that $S(\mathfrak{g})^G$ is isomorphic to $S(\mathfrak{g}^*)^G$, that is, to the ring of $G$-invariant polynomial functions of the entries of a matrix $X \in \mathfrak{gl}_n(\mathbb{C})$. Then deduce from Problem 1 that $Z(\mathfrak{A}(\mathfrak{gl}_n))$ is a polynomial ring on $n$ generators.

3. (a) Let $X$ and $Y$ be generic strictly upper triangular 4-by-4 matrices; that is, matrices whose non-zero entries are indeterminates $x_{ij}$, $y_{ij}$. Find the entries of the matrix $\log(\exp X \exp Y)$, as polynomials in the $x_{ij}$ and $y_{ij}$, by direct matrix computation.

(b) Use part (a) to recover the coefficients of the terms of degree 3 in the Baker-Campbell-Hausdorff series.

(c) Can this computation be generalized to determine the coefficient of any term of the BCH formula?

4. (a) Show that the classicifation of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, and the fact that all finite dimensional representations are semisimple, imply corresponding results for $\mathfrak{sl}_2(\mathbb{R})$.

(b) Let $G$ be the simply connected covering group of $SL_2(\mathbb{R})$. Use (a) to show that every finite-dimensional representation of $G$ factors through a representation of $SL_2(\mathbb{R})$, and hence that $G$ has no faithful finite-dimensional linear representation.

5. Let $G$ be a Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$. Let $V$ be a continuous representation of $G$, *i.e.*, the action is a Lie group homomorphism $G \to GL(V)$. Then the induced homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ makes $V$ a $\mathfrak{g}$ module. Prove that if $W \subseteq V$ is a $\mathfrak{g}$ submodule, and $G$ is connected, then $W$ is a $G$ submodule of $V$, that is, $gW \subseteq W$ for all $g \in G$.

6. Let $E$ be the quotient of the abelian Lie group $(\mathbb{C}, +)$ by a full rank lattice, that is, a discrete subgroup isomorphic to $\mathbb{Z}^2$. Then $E$ is a compact complex Lie group (in algebraic geometry, these groups are the *elliptic curves*). Show that $E$ has no faithful finite-dimensional linear representation as a complex Lie group, that is, there is no injective homomorphism of complex Lie groups $E \to GL_n(\mathbb{C})$ for any $n$. Hint: start by using Problem 5 to show that the only irreducible representation of $E$ is trivial.

7. (a) Does there exist a non-zero finite-dimensional Lie algebra $\mathfrak{g}$ such that the only irreducible finite-dimensional representation of $\mathfrak{g}$ is the trivial representation?

(b) Show that if $\mathfrak{g}$ is a simple infinite-dimensional Lie algebra, then its only irreducible finite-dimensional representation is the trivial representation.

(c) Construct an example of a simple infinite-dimensional Lie algebra.

8. A *real form* of a connected complex Lie group $G$ is a connected real Lie subgroup $H$ such that $\mathfrak{g} = \mathrm{Lie}(G)$ is the complexification of $\mathfrak{h} = \mathrm{Lie}(H)$, that is, $\dim_{\mathbb{R}}(\mathfrak{h}) = \dim_{\mathbb{C}}(\mathfrak{g})$ and $\mathfrak{g} = \mathbb{C}\mathfrak{h}$.

Show that the unitary group $U(n)$ and the real general linear group $GL_n(\mathbb{R})$ are real forms of $GL_n(\mathbb{C})$, with non-isomorphic real Lie algebras for $n > 1$. Hint: show that the endomorphism $\mathrm{ad}(x)$ of $\mathfrak{gl}_n(\mathbb{C})$ is has purely imaginary eigenvalues for every $x \in \mathfrak{u}_n = \mathrm{Lie}(U(n))$.