# Math 261A: Lie Groups, Fall 2015
# Problem Set 1

Problems from Varadarajan:

Chapter 1, Exercise 2.

Chapter 2, Exercises 2, 3, 4, 9(a), 11, 12(b,c), 16 (assuming as in the hint that a Lie group with Lie algebra $\mathfrak{g}/\mathfrak{h}$ exists), 18, 19, 20, 21, 22, 27, 39.

Other problems:

1. (a) For any object $X$ in a category $C$, the functor $h_X = \mathrm{Hom}_C(-, X)$ from $C^{\mathrm{op}}$ (the category $C$ with arrows reversed) to **Sets** is called the functor *represented by* $X$. Suppose $C$ is a category with finite products. Show that to give an object $G$ the structure of a group object in $C$ is equivalent to giving a functor $g_G \colon C^{\mathrm{op}} \to \mathbf{Groups}$ such that $h_G = f \circ g_G$, where $f \colon \mathbf{Groups} \to \mathbf{Sets}$ is the 'forgetful functor' sending a group to its underlying set.

(b) Show that to give an action of a group object $G$ in $C$ on an object $X$ in $C$ (that is, an arrow $G \times X \to X$ such that suitable diagrams corresponding to the usual definition of a group action commute) is equivalent to giving a an action of the group $g_G(T)$ on the set $h_X(T)$, for every object $T$ in $C$, which is functorial in $T$.

(c) Show that when $C$ is the category of topological spaces, or of smooth, analytic or holomorphic manifolds, $G$ is a group object if and only if its underlying set is a group, in such a way that that the group law and the map $g \mapsto g^{-1}$ are morphisms in the category, that is, continuous, smooth, analytic or holomorphic maps.

(d) Show that in the categories of spaces in part (c) an action of a group object $G$ on an object $X$ is the same as a group action of the underlying set of $G$ on the underlying set of $X$, such that the action map $G \times X \to X$ is a morphism in the category.

2. Prove that if $G$ is a manifold and a group such that the group operation is smooth (resp. analytic, holomorphic), then the map $i \colon g \mapsto g^{-1}$ is automatically smooth (resp. analytic, holomorphic). Hint: show that the group operation $\mu$ is a submersion, and deduce that the graph of $i$ is a regularly embedded submanifold of $G \times G$.

3. (a) Show that the tangent space $T_pX$ of a real analytic manifold $X$ at a point $p$ is canonically identified with the tangent space at $p$ of $X$ considered as a real smooth manifold.

(b) Show that if $X$ is a complex holomorphic manifold, the tangent space $T_pX$, when regarded as a real vector space, is canonically identified with the tangent space at $p$ of $X$ considered as a real analytic manifold, of twice its complex dimension.

4. (a) Construct an isomorphism of Lie algebras $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C}) \times \mathfrak{so}_3(\mathbb{C})$.

(b) Construct a corresponding isogeny of Lie groups $SO_4(\mathbb{C}) \to SO_3(\mathbb{C}) \times SO_3(\mathbb{C})$ and show that its kernel is $\{\pm I_4\}$.

5. (a) Show that the simply connected covering space of $SL_2(\mathbb{R})$ can be described as the set $\widehat{SL_2(\mathbb{R})}$ of pairs $(g, \theta)$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, and $\theta \in \mathbb{R}$ is a value of $\arg(a + ib)$, the covering map being given by $(g, \theta) \mapsto g$.

(b) Write down the group law on $\widehat{SL_2(\mathbb{R})}$ explicitly in terms of $g$ and $\theta$. (You will probably find this impossible to express as a simple closed formula.)

6. Let $G$ be the group of Euclidean motions of $\mathbb{R}^n$, that is, the semidirect product $SO(n,\mathbb{R}) \ltimes \mathbb{R}^n$, where $\mathbb{R}^n$ acts on itself by translations. Describe the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and the exponential map $\exp\colon \mathfrak{g} \to G$.

7. Prove that for any Lie group $G$, if $x, y \in \mathrm{Lie}(G)$ satisfy $[x,y] = 0$, then $\exp(x+y) = \exp(x)\exp(y)$. (You might do this using either the Baker-Campbell-Hausdoff formula or Chevalley's subgroup theorem. If you use BCH you will need to deal with the fact that $x, y$ are not assumed to lie in a domain on which the formula converges.)

8. Calculate explicitly, in terms of matrix coordinates, the left and right invariant vector fields $\lambda_x$ and $\rho_x$ on $G = GL_n$ with value $x \in T_eG = \mathfrak{gl}_n$ at the identity. Then verify directly that $[\lambda_x, \lambda_y] = \lambda_{[x,y]}$, where $[x,y]$ is matrix commutator, and that $[\rho_x, \rho_y] = -\rho_{[x,y]}$.

9. In the situation of Problem 8, express the differential $(d\exp)_x$ of the exponential map at $x \in \mathfrak{gl}_n$ explicitly in terms of matrix coordinates. Use this to find the vector field $\xi_y$ on $\mathfrak{gl}_n$ which is related via exp to $\lambda_y$, and verify that the result agrees with the formula

$$\xi_y(x) = \frac{\mathrm{ad}\, x}{1 - e^{-\mathrm{ad}\, x}}\, y.$$

On what subset of $\mathfrak{g}_n$ is $\xi_y$ defined for all $y$? How is this related to the locus where $d\exp$ is singular, and why?

10. Let $T$ be the tensor algebra over $\mathbb{Q}$ on generators $X, Y$ and let $F$ be the Lie subalgebra of $T$ generated by $X$ and $Y$, with commutator in $T$ as Lie bracket. Note that $F$ is a graded subspace of $T$, that is, $F$ is the direct sum of its degree components $F_n = F \cap T_n$, since if $a \in T_m$, $b \in T_n$, then $[a,b] = ab - ba \in T_{m+n}$.

For $q \in T$, let $\Theta(q)$ be the operator on $F$ given by subsituting $(\mathrm{ad}\, X)$ for $X$ and $(\mathrm{ad}\, Y)$ for $Y$ in $q$. Define a $\mathbb{Q}$-linear map $\Psi\colon T \to F$ by $\Psi(1) = 0$ and $\Psi(qZ) = \Theta(q)Z$ for $Z = X, Y$.

Explicitly, given a tensor monomial $Z_1 \cdots Z_n \in T$, where each $Z_i$ is either $X$ or $Y$, we have

$$\Psi(Z_1 \cdots Z_n) = (\mathrm{ad}\, Z_1) \cdots (\mathrm{ad}\, Z_{n-1}) Z_n.$$

(a) Using the fact that $\mathrm{ad}\colon F \to \mathrm{End}_{\mathbb{Q}}(F)$ is a Lie algebra homomorphism, conclude that $\Theta(q) = \mathrm{ad}\, q$ if $q \in F$.

(b) Show that $F_n = (\mathrm{ad}\, X)F_{n-1} + (\mathrm{ad}\, Y)F_{n-1}$. In other words, Lie bracket monomials $(\mathrm{ad}\, Z_1)\cdots(\mathrm{ad}\, Z_{n-1})Z_n$ span $F$ (but are not linearly independent).

(c) Prove that $\Psi(p) = np$ if $p \in F_n$, by induction on $n$, using (a) and (b).

(d) Let $B(X,Y) = X + Y + \frac{1}{2}[X,Y] + \cdots$ be the Baker-Campbell-Hausdorff series. We can consider $B(tX, tY) = (X+Y)t + \frac{1}{2}[X,Y]t^2 + \cdots$ as a formal power series in $t$ whose coefficient of $t^n$ belongs to $F_n$ and thus to $T_n$. As such, it is just the formal logarithm $\log(e^{tX}e^{tY})$, where $\log(1+\phi(t)) = \sum_{k=1}^{\infty}(-1)^{k-1}\phi(t)^k/k$ for any series $\phi(t)$ with coefficients in $T$ and zero constant term.

Use this to obtain the explicit formula, due to Dynkin,

$$B(tX,tY) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{p_1+q_1\geq 1,\ldots,p_k+q_k\geq 1} \frac{\Psi(X^{p_1}Y^{q_1}\cdots X^{p_k}Y^{q_k})}{p_1!q_1!\cdots p_k!q_k!(\sum p_i + \sum q_i)} t^{\sum p_i + \sum q_i}$$

11. (a) Let $\phi\colon S^2 \to \mathbb{CP}^1$ be the map given by stereographic projection from the north pole of $S^2$ to the complex plane $\mathbb{C}$, with $\phi$ mapping the south pole to 0, the equator to the unit circle $\{|z| = 1\}$, and the north pole to $\infty$. Verify that $\phi$ is an isometry between the standard angle metric on $S^2$ and the Fubini-Study metric on $\mathbb{CP}^1$ given by $d(\overline{x}, \overline{y}) = 2\cos^{-1}|(x, y)|$, where $x, y \in \mathbb{C}^2$ are unit vectors.

(b) Work out the resulting Lie group homomorphism $\psi\colon U(2) \to SO(3)$ in explicit coordinates, *i.e.*, find the entries of the $3\times 3$ real orthogonal matrix $\psi(A)$ in terms of the entries of the $2 \times 2$ complex unitary matrix $A$.

12. Construct an isomorphism of $GL(n, \mathbb{C})$ (as a Lie group and an algebraic group) with a closed subgroup of $SL(n + 1, \mathbb{C})$.

13. Show that the map $\mathbb{C}^* \times SL(n, \mathbb{C}) \to GL(n, \mathbb{C})$ given by $(z, g) \mapsto zg$ is a surjective homomorphism of Lie and algebraic groups, find its kernel, and describe the corresponding homomorphism of Lie algebras.

14. The *free Lie algebra* on generators $X_i$ over a field $k$ is a Lie algebra $F$ over $k$ with generators $X_i$ and only the relations that follow from the Lie algebra axioms. More precisely, $F$, together with its distinguished elements $X_i$, is characterized by the property that for every Lie algebra $\mathfrak{g}$ over $k$ and system of elements $x_i \in \mathfrak{g}$, there is a unique Lie algebra homomorphism $F \to \mathfrak{g}$ sending $X_i$ to $x_i$.

(a) Show that an $F$ module is just a vector space $V$ together with arbitrary endomorphisms $\xi_i = \rho(X_i)$.

(b) Deduce that the universal enveloping algebra of $F$ is the tensor algebra $T$ over $k$ on the generators $X_i$. (This is slightly subtle. It is clear from (a) that the associative algebras $T$ and $\mathfrak{A}(F)$ have canonically equivalent categories of modules, but not entirely obvious that this implies that $T$ and $\mathfrak{A}(F)$ are isomorphic.)

(c) Using Poincaré-Birkhoff-Witt, deduce that $F$ is isomorphic to the Lie subalgebra of $T$ generated by the elements $X_i$, where we regard $T$ as a Lie algebra with commutator as the Lie bracket.

(d) Assume now that the set of generators $X_i$ is a finite set $\{X_1, \ldots, X_n\}$, so that the graded algebras $T$ and $F$ have finite dimension in each degree. In particular, $t_d = \dim(T_d) = n^d$ is the number of words (or tensor monomials) of length $d$ in the $n$ letters $X_i$, with generating function

$$\sum_d t_d\, z^d = \frac{1}{1 - nz}.$$

Show that $f_d = \dim(F_d)$ is characterized by the identity

$$\prod_d \frac{1}{(1 - z^d)^{f_d}} = \frac{1}{1 - nz}.$$

(e) Derive the explicit formula $f_d = (1/d)\sum_{k|d} \mu(d/k)\, n^k$. Here $\mu(m)$ is the classical Möbius function, equal to $(-1)^r$ if $m$ is a product of $r$ distinct primes, or 0 if $m$ is divisible by a square, which is characterized by the Möbius inversion formula $a_d = \sum_{k|d} \mu(k/d) b_k$ if $b_d = \sum_{k|d} a_k$, for any sequence $a_1, a_2, \ldots$.

(f) A word $w$ in the letters $X_1, \ldots, X_n$ is *aperiodic* if all rotations of $w$ are distinct. Show that $f_d$ is equal to the number of rotation classees of non-empty aperiodic words of length $d$ in $n$ letters.

(g) A *Lyndon word* is a non-empty aperiodic word which is lexicographically least in its rotation class. Thus $f_d$ is the number of Lyndon words of length $d$ in $n$ letters. Show that every Lyndon word $w$ of length $d > 1$ can be factored (not necessarily uniquely) as $w = uv$, where $u$ and $v$ are Lyndon. Hint: it works to take for $v$ the right factor such that $vu$ is the lexicographically least rotation of $w$ other than $w$ itself.

(h) Fix one Lyndon factorization $w = uv$ for each Lyndon word $w$ of length $d > 1$, and define a Lie bracket monomial $[w] \in F$ inductively by $[w] = X_i$ if $w = X_i$, otherwise $[w] = [[u], [v]]$, where $w = uv$ is the chosen factorization. Show that the lexicographically least term of $[w]$, considered as an element of $T$, is $w$.

(i) Deduce that the Lie bracket monomials $[w]$ for all Lyndon words $w$ form a basis of the free Lie algebra $F$ (for any given choice of the factorizations $w = uv$).

(j) Show that if $n$ is a power of a prime, so there exists a finite field $\mathbb{F}$ of order $n$, then $f_d$ is equal to the number of distinct monic irreducible polynomials $g(x)$ of degree $d$ over $\mathbb{F}$. Is this purely a numerical coincidence, or can some deeper connection with the free Lie algebra be found?

15. Prove that if $\mathfrak{g}$ is a solvable Lie algebra over $\mathbb{R}$, then every finite-dimensional irreducible $\mathfrak{g}$ module has dimension at most 2.

16. Construct an example of a solvable Lie algebra $\mathfrak{g}$ over a field of characteristic 2 such that the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is not nilpotent. Hint: start with the the 2-dimensional non-nil module $V$ for the 3-dimensional Heisenberg algebra $\mathfrak{h}$, and form the semidirect product of $\mathfrak{h}$ with $V$, regarded as an abelian Lie algebra.