VARIETIES AS SCHEMES

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0.1. Classical algebraic geometry is the study of algebraic varieties, meaning spaces that can be described locally as solution sets of polynomial equations over an algebraically closed field, such as the complex numbers $\mathbb{C}$.

In the 1960's, Grothendieck reformulated the foundations of the subject as the study of *schemes*. The language of schemes has since come to be universally accepted because of its advantages in both technical simplicity and conceptual clarity over older theories of varieties. At its heart, however, algebraic geometry is still primarily the study of algebraic varieties. The most important examples of schemes either arise directly from varieties, or are closely related to them. In order to apply scheme theory to varieties, or indeed to get any real idea of what schemes are are all about, one must first understand *how classical varieties are schemes.*

0.2. The purpose of these notes is to state precisely and prove the equivalence

$$\text{classical algebraic varieties over } k = \text{reduced algebraic schemes over } k,$$

where k is an algebraically closed field.

The basic outlines of this equivalence are simple to describe. We can turn any variety X into a scheme Y over k by 'soberizing' its underlying topological space, and regarding the sheaf of regular functions on X as an abstract sheaf of rings $\mathcal{O}_Y$ on Y, forgetting its realization as a sheaf of functions. Algebraic maps $X' \to X$ between varieties are then in one-to-one correspondence with k-morphisms $Y' \to Y$ between the schemes constructed from them. In other words, the construction gives an equivalence of categories from varieties to a suitable full subcategory of the category of schemes over k. This subcategory turns out to consist of the *reduced algebraic schemes over* k. Here, and more generally, an *algebraic scheme* means a scheme locally of finite type over a field.

The inverse construction is also not difficult. Given a reduced algebraic scheme Y over an algebraically closed field k, the underlying space of the associated variety X is the subspace Y_{cl} of closed points of Y. The closed points of Y are also its k-points, enabling us to identify the structure sheaf $\mathcal{O}_Y$ with a sheaf of functions on X by evaluating its sections in the residue field at each point.

0.3. To fully establish the equivalence in §0.2 we need to verify a number of things. For instance, we need to see that the construction starting with a variety gives a scheme, reduced and algebraic over k, that the construction starting with such a scheme gives a variety, that the constructions are inverse to one another, and that they are functorial, sending morphisms of varieties to morphisms of k-schemes and inversely.

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We also want to make the equivalence more explicit for affine varieties and schemes by showing that the affine variety X with coordinate ring $R = \mathcal{O}(X)$ corresponds to the affine scheme $\mathrm{Spec}(R)$. Having the affine case in this explicit form makes the equivalence effective as a tool for working with varieties using language and techniques from the theory of schemes. The affine case also provides the main ingredient in the proof of the equivalence, with the rest following from affine coverings of more general varieties and schemes.

We will not try to give the shortest possible proof of the equivalence, but rather try to follow a line of reasoning that (we hope) is conceptually transparent, more geometric than algebraic, and natural from a scheme-theoretic point of view.

In order to have anything to prove, we will need to know a little bit about classical varieties, independent of their interpretation as schemes. To this end we give in §1 a brief introduction to varieties, using the definition due to Serre in *Faisceaux Algébriques Cohérents.*

We will develop here only a minimal portion of the theory of varieties, our attitude being that everything else about varieties should be understood via the interpretation of varieties as schemes. An example is Hilbert's Nullstellensatz, which in traditional approaches appears as one of the first fundamental theorems on varieties, usually with a proof based on some not very geometric-looking theorem about fields. However, its analog for schemes (that closed subsets of $\mathrm{Spec}(R)$ are in one-to-one correspondence with radical ideals in R) is an easy general theorem of commutative algebra. The version for classical varieties then follows from their interpretation as schemes.

We presume the reader to be familiar with some basics of scheme theory, including the definitions of a scheme and a morphism of schemes, and the notions of scheme and morphism over a base scheme S, abbreviating these as usual to "k-scheme" and "k-morphism" when $S = \mathrm{Spec}(k)$. We take as known the theorem that the ring of global functions $\mathcal{O}_Y(Y)$ on an affine scheme $Y = \mathrm{Spec}(R)$ is equal to R, and the characterization of morphisms $X \to \mathrm{Spec}(R)$ as corresponding to ring homomorphisms $R \to \mathcal{O}_X(X)$. We also take as known the construction of fiber products of schemes, and the theorem that the scheme-theoretic fibers of a morphism of schemes are homeomorphic to the topological fibers.

Some other more specialized aspects of scheme theory will be developed here as needed, such as the theory of Jacobson schemes and morphisms locally of finite type. Our approach generally follows that of Grothendieck in *Éléments de géométrie algébrique* (EGA), particularly EGA IV, Ch. 10 and (for sober spaces) the 1971 revised edition of EGA I.

0.4. Conventions: all rings are commutative, with unit. All ring homomorphisms are unital.

1. Classical varieties

1.1. Let k be an algebraically closed field. A classical *affine algebraic variety* X is the solution locus in k^n of a system of polynomial equations. Rewriting equations $g = h$ as $g - h = 0$, we can also say that X is the zero locus $V(F) = \{a \in k^n \mid f(a) = 0 \text{ for all } f \in F\}$ of a set of polynomials $F \subseteq k[\mathbf{x}] = k[x_1, \ldots, x_n]$.

1.2. The identities

$$\emptyset = V(\{1\}), \quad k^n = V(\emptyset), \quad V(F) \cup V(G) = V(F \cdot G), \quad \bigcap_\alpha V(F_\alpha) = V(\bigcup_\alpha F_\alpha)$$

show that the affine varieties $X \subseteq k^n$ are the closed subsets of a topology on k^n, called the *Zariski topology*. For this reason, we also refer to an affine variety $X \subseteq k^n$ as a *closed subvariety* of k^n. The Zariski topology on X is defined to be its subspace topology in k^n. The closed subsets of X in the Zariski topology are thus the closed subvarieties $Y \subseteq k^n$ that are contained in X.

1.3. Let Y be any subset of k^n. The set $\mathcal{I}(Y) = \{f \mid Y \subseteq V(f)\}$ of polynomials vanishing on Y is an ideal in the polynomial ring $k[\mathbf{x}] = k[x_1, \ldots, x_n]$. Since $F = \mathcal{I}(Y)$ is the largest subset of $k[\mathbf{x}]$ such that $Y \subseteq V(F)$, we see that $X = V(\mathcal{I}(Y))$ is the smallest closed subvariety containing Y, that is, the closure of Y in the Zariski topology. In particular, for every closed subvariety $X \subseteq k^n$, we have $X = V(\mathcal{I}(X))$. This shows that the correspondence

$$X \mapsto \mathcal{I}(X), \quad I \mapsto V(I)$$

is an order-reversing bijection between closed subvarieties $X \subseteq k^n$ and ideals $I \subseteq k[\mathbf{x}]$ of the form $I = \mathcal{I}(X)$. According to Hilbert's Nullstellensatz, the latter are exactly the radical ideals $I = \sqrt{I}$ in $k[\mathbf{x}]$. We will not make any use of the Nullstellensatz at this stage. We will prove it later (Lemma 8.5), as part of the equivalence between varieties and schemes.

1.4. For any closed subvariety $X \subseteq k^n$, we define $\mathcal{O}(X) = k[\mathbf{x}]/\mathcal{I}(X)$. Since $\mathcal{I}(X)$ is by definition the kernel of the evaluation homomorphism from $k[\mathbf{x}]$ to k-valued functions on X, we can identify $\mathcal{O}(X)$ with the ring of functions f on X given by polynomials in the coordinates $x_1, \ldots, x_n$. We call $\mathcal{O}(X)$ the ring of *polynomial functions on* X or the *coordinate ring* of X.

Since $\mathcal{O}(X)$ is a ring of functions on X, we can define the vanishing locus $V(F) \subseteq X$ for any subset $F \subseteq \mathcal{O}(X)$, and the ideal $\mathcal{I}(Y) \subseteq \mathcal{O}(X)$ for any $Y \subseteq X$. Then $V(F) \subseteq X$ is always closed, and every closed subvariety $Y \subseteq X$ has the form $Y = V(I)$, where $I = \mathcal{I}(Y)$. These are corollaries to the corresponding facts in the case $X = k^n$.

1.5. For every $f \in \mathcal{O}(X)$, we define an open subset $X_f = X - V(f)$ of X. We have $f(a) \neq 0$ for all $a \in X_f$, so the function $1/f$ is defined on X_f. Any open subset $U = X - V(F)$ of X is the union of the subsets X_f for $f \in F$. Hence the sets X_f form a base of open subsets for the Zariski topology on X. We also have $X_f \cap X_g = X_{fg}$.

1.6. Given any topological space X, let $\mathcal{F}_X^k$ denote the sheaf of all k-valued functions on open subsets of X.

We define the *sheaf of regular functions* $\mathcal{O}_X$ on a classical affine variety X to be the subsheaf $\mathcal{O}_X \subseteq \mathcal{F}_X^k$ consisting of functions locally of the form $f = g/h$ on X_h. More precisely, for each open $U \subseteq X$, a function $f\colon U \to k$ belongs to $\mathcal{O}_X(U)$ if and only if every point $a \in U$ has a neighborhood $a \in X_h \subseteq U$ such that $f = g/h$ on X_h, for some $g, h \in \mathcal{O}(X)$ such that $h(a) \neq 0$. The local nature of the definition implies that $\mathcal{O}_X$ is a subsheaf of $\mathcal{F}_X^k$ and not just a sub-presheaf.

By construction, any $f \in \mathcal{O}(X)$, when restricted to X_f, is an invertible element of $\mathcal{O}_X(X_f)$. The restriction homomorphism $\mathcal{O}(X) \to \mathcal{O}_X(X_f)$ therefore extends uniquely to a k-algebra homomorphism $\rho_f\colon \mathcal{O}(X)_f \to \mathcal{O}_X(X_f)$. In fact, ρ_f is an isomorphism, with $\mathcal{O}(X) \cong \mathcal{O}_X(X)$ as a special case. We will not prove or use this now, but will deduce it later (§8.9) from the equivalence between classical affine varieties and schemes.

1.7. Any classical affine variety $X \subseteq k^n$ can be recovered, as a space equipped with a sheaf of rings of k-valued functions $\mathcal{O}_X$, from its coordinate ring $R = \mathcal{O}(X) = k[x_1, \ldots, x_n]/\mathcal{I}(X)$. We now explain the procedure for doing this, taking particular note of how the geometric data for $(X, \mathcal{O}_X)$ are related to the affine scheme $Y = \operatorname{Spec}(R)$ over k.

To each point $a \in X$ there corresponds an evaluation homomorphism $\mathrm{ev}_a \colon R \to k$ given by $\mathrm{ev}_a(f) = f(a)$. Since ev_a is a k-algebra homomorphism, it is surjective, and is determined by its kernel, the maximal ideal $\mathfrak{m}_a = \mathcal{I}(\{a\}) \subseteq R$. Every k-algebra homomorphism $\varphi \colon R \to k$ is of the form $\varphi = \mathrm{ev}_a$ for a unique point $a \in X$, namely, $a = (a_1, \ldots, a_n)$, where $a_i = \varphi(x_i)$.

Thus X is in canonical bijection with the set of k-algebra homomorphisms from R to k, or in other words, with the set $\underline{Y}(k)$ of k-points of the k-scheme $Y = \operatorname{Spec}(R)$, and also with the set of those maximal ideals $\mathfrak{m} \subseteq R$, or closed points of Y, such that $R/\mathfrak{m} = k$. Later (§8.3) we will see that the residue field $R/\mathfrak{m}$ at *every* closed point of Y is equal to k, but for now we will not assume that we know this.

In terms of the bijection $X \to \underline{Y}(k)$, the closed subsets $V(I) \subseteq X$ are determined by $a \in V(I) \Leftrightarrow I \subseteq \mathfrak{m}_a$. In other words, the topology on X is the same as its subspace topology in Y, when we identify X with a set of closed points in Y.

Having identified the points of X with homomorphisms $\mathrm{ev}_a \colon R \to k$, we can recover the function corresponding to any $f \in R$ from the formula $f(a) = \mathrm{ev}_a(f)$. We then recover the sheaf $\mathcal{O}_X$ in the same way that it was originally defined, that is, $\mathcal{O}_X(U)$ is the set of functions $f \colon U \to k$ locally of the form $f = g/h$ on X_h, where $g, h \in R$.

This description of the sheaf of functions $\mathcal{O}_X$ can be expressed in terms of the structure sheaf $\mathcal{O}_Y$ of Y, in the following manner. Identifying X with a subspace of closed points in Y, we have $X_h = X \cap Y_h$. We can evaluate any $f \in \mathcal{O}_Y(Y_h) = R_h$ in the residue field k at each point $a \in X_h$ to get a k-valued function on X_h. This gives a homomorphism $i^\flat \colon \mathcal{O}_Y \to i_* \mathcal{F}^k_X$ of sheaves of rings from $\mathcal{O}_Y$ to the sheaf of k-valued functions on open subsets of $i(X)$, where $i \colon X \hookrightarrow Y$ is the identification map. The definition of $\mathcal{O}_X$ characterizes it as the subsheaf of $\mathcal{F}^k_X$ such that $i_* \mathcal{O}_X$ is the image sheaf of the homomorphism $i^\flat$.

Remark. One advantage to viewing a classical affine variety X as an object constructed from its coordinate ring R is that it gives a coordinate-free description of X. Each particular embedding of X as a closed subvariety in a coordinate space k^n corresponds to the choice of a system of generators $x_1, \ldots, x_n$ for R as a k-algebra.

1.8. We now define a general *classical algebraic variety over* k to be a topological space X, equipped with a sheaf of rings of k-valued functions $\mathcal{O}_X \subseteq \mathcal{F}^k_X$, such that there exists a covering of X by open subsets U for which $(U, \mathcal{O}_X|U)$ is isomorphic to a classical affine variety. This is essentially the definition in Serre, *Faisceaux Algébriques Cohérents.*

A fundamental non-affine example is given by the classical n-dimensional projective space $\mathbb{P}^n$ over k, with its covering by open affine spaces $U_0, \ldots, U_n$. More generally, if $X \subseteq \mathbb{P}^n$ is a projective variety, that is, the zero locus of a set of homogeneous polynomials in projective coordinates, then each $X \cap U_i$ is an affine variety, closed in $U_i \cong k^n$. This makes X a classical variety and a closed subvariety of $\mathbb{P}^n$.

A *morphism* of classical algebraic varieties is a continuous map $f \colon X \to Y$ such that the canonical map $f^\flat \colon \mathcal{F}^k_Y \to f_* \mathcal{F}^k_X$ given by composition with f carries the subsheaf $\mathcal{O}_Y \subseteq \mathcal{F}^k_Y$ into $f_* \mathcal{O}_X \subseteq f_* \mathcal{F}^k_X$. More concretely, this means that regular functions on any open $U \subseteq Y$, when composed with f, give regular functions on $f^{-1}(U)$.

1.9. Suppose X and Y are classical affine varieties. Given any k-algebra homomorphism $\varphi\colon \mathcal{O}(Y) \to \mathcal{O}(X)$, the identification of points with evaluation homomorphisms, as in §1.7, yields a unique map $f\colon X \to Y$ such that $\mathrm{ev}_{f(a)} = \mathrm{ev}_a \circ \varphi$ for all $a \in X$. To describe f in terms of coordinates, suppose that $X \subseteq k^m$ and $Y \in k^n$. Then for every point $a \in X$, the coordinates $(y_1, \ldots, y_n)$ of $f(a)$ are polynomial functions in the coordinates $(x_1, \ldots, x_m)$ of a, given by $\varphi(y_i) \in \mathcal{O}(X)$.

It is easy to see that f is continuous in the Zariski topology. If $h \in \mathcal{O}(Y)$ is a polynomial function on Y, then $h \circ f = \varphi(h)$, so composition with f carries $\mathcal{O}(Y)$ into $\mathcal{O}(X)$. It follows from this and the definition of the sheaves of regular functions that $f^\flat(\mathcal{O}_Y) \subseteq \mathcal{O}_X$. Hence f is a morphism of classical varieties.

Conversely, given the identity $\mathcal{O}(X) = \mathcal{O}_X(X)$ mentioned in §1.6, if $f\colon X \to Y$ is a morphism between classical affine varieties, then we must have $f^\flat(\mathcal{O}(Y)) \subseteq \mathcal{O}_X(X) = \mathcal{O}(X)$. This provides a k-algebra homomorphism $\varphi\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ from which f arises as above. In other words, every morphism between classical affine varieties is given in terms of coordinates of points by a polynomial map.

We will only use this characterization of morphisms between classical affine varieties after proving the identity $\mathcal{O}(X) = \mathcal{O}_X(X)$ in §8.9.

1.10. If $X \subseteq k^n$ is a classical affine variety and $f \in R = \mathcal{O}(X)$ is a polynomial function on X, then the open subspace X_f, with its sheaf of regular functions defined by restriction from X, is itself isomorphic to an affine variety.

We will prove this in §8.9 using the construction of X from the scheme $Y = \mathrm{Spec}(R)$. It is also not too hard, and may serve as an illustrative exercise for the reader, to prove it directly from the definitions. Hint: if $X \subseteq k^n$, then X_f is isomorphic to a variety $Z \in k^{n+1}$ with an extra coordinate z and an additional equation $zf(x) = 1$.

This result implies that open affine subsets of a classical variety X not only cover X, but form a base of the topology on X. It follows in particular that every open subspace of a classical variety is a classical variety.

2. Sober spaces

2.1. What kind of topological space X can be the underlying space of a scheme? In general, such spaces satisfy only the weakest of the standard separation axioms (X is T_0), but they have another property: X is a *sober space*, which means that every irreducible closed subset of X is the closure of a unique point.

In this section we discuss sober spaces, the soberization functor, and quasi-homeomorphisms, and we prove that schemes are sober. We develop a little more of the theory than we will actually use, because a thorough understanding of these concepts is helpful for getting a good intuitive grasp of the correspondence between varieties and schemes, as well as of the general nature of schemes as topological spaces.

2.2. A topological space Z is *irreducible* if Z is non-empty, and Z is not a union of two proper closed subsets (hence not a union of any finite number of proper closed subsets).

Other equivalent ways to formulate the condition that a non-empty space Z is irreducible are (a) every intersection $U_1 \cap U_2$ of two non-empty open subsets of Z is non-empty; or (b) every non-empty open subset of Z is dense in Z.

2.3. *Proposition.*
(i) If $f\colon Z \to X$ is continuous and Z is irreducible, then $f(Z)$ is irreducible.
(ii) If $Z \subseteq X$ is an irreducible subspace, then the closure $\overline{Z}$ is irreducible.
(iii) Every non-empty open subset of an irreducible space is irreducible.

The proof is easy. Note, in particular, that the closure $\overline{\{x\}}$ of any point in any space X is always irreducible.

2.4. *Proposition–Definition.* Every space X is the union of its maximal irreducible closed subspaces, called the *irreducible components* of X.

Proof. If (Z_α) is a non-empty family of irreducible subspaces of X, totally ordered by inclusion, then $Z = \bigcup_\alpha Z_\alpha$ is irreducible. For, if $Z \subseteq Y_1 \cup Y_2$, where Y_1, Y_2 are closed in X, then each Z_α, being irreducible, is contained in Y_1 or in Y_2. It follows that for at least one of the indices $i = 1, 2$, the Z_β such that $Z_\beta \subseteq Y_i$ are co-final in the chain (Z_α). Then $Z \subseteq Y_i$.

It follows by Zorn's lemma that every irreducible subspace of X is contained in a maximal irreducible subspace. In particular, $\{x\}$ is contained in some maximal irreducible subspace $Z \subseteq X$, for every $x \in X$. Thus X is the union of its maximal irreducible subspaces, and every such subspace is closed, by Proposition 2.3, (ii). $\square$

2.5. *Definition.* A topological space X is *sober* if every irreducible closed subset Z of X is the closure $Z = \overline{\{z\}}$ of a unique point z.

As a trivial example, any Hausdorff space is sober, since its only irreducible subspaces are the one-point sets $\{x\}$. More importantly, as we will see next, every scheme is sober.

Recall that, by definition, a space X is T_0 if for every two distinct points of X, there is an open subset of X that contains exactly one of them. An equivalent condition is that $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$ for all $x, y \in X$. In particular, every sober space is T_0.

2.6. *Lemma.* If X has a covering by sober open subspaces U_α, then X is sober.

Proof. It is easy to see that the open covering by T_0 spaces U_α implies that X is T_0.

Let $Z \subseteq X$ be an irreducible closed subset. For each α such that $Z \cap U_\alpha \neq \emptyset$, we have $Z \cap U_\alpha = \overline{\{z_\alpha\}} \cap U_\alpha$ for a unique $z_\alpha \in U_\alpha$, since U_α is sober. If $Z \cap U_\alpha \neq \emptyset$ and $Z \cap U_\beta \neq \emptyset$, then $Z \cap U_\alpha \cap U_\beta \neq \emptyset$, since Z is irreducible. Since $\{z_\alpha\}$ is dense in $Z \cap U_\alpha$, it follows that $z_\alpha \in Z \cap U_\beta \subseteq \overline{\{z_\beta\}}$. By symmetry, we also have $z_\beta \in \overline{\{z_\alpha\}}$, hence $\overline{\{z_\alpha\}} = \overline{\{z_\beta\}}$. Since X is T_0, $z_\alpha = z_\beta$. This shows that there is a single point z such that

$$Z \cap U_\alpha = \overline{\{z\}} \cap U_\alpha$$

for all α such that $Z \cap U_\alpha \neq \emptyset$. The same identity holds trivially if $Z \cap U_\alpha = \emptyset$, since Z, being closed, contains $\overline{\{z\}}$. Since (U_α) is an open cover of X, it follows that $Z = \overline{\{z\}}$, and since X is T_0, z is unique. $\square$

2.7. *Proposition.* Every scheme is sober.

Proof. By Lemma 2.6, we can reduce to the case of an affine scheme $X = \operatorname{Spec}(R)$. The irreducible closed subsets of X are then the sets $Z = V(P)$, where P is a prime ideal in R. The closure of any subset $Y \subseteq \operatorname{Spec}(R)$ is given by $\overline{Y} = V(I)$, where $I = \cap_{Q \in Y} Q$. In particular, $\overline{\{P\}} = V(P) = Z$. If we also had $\overline{\{P'\}} = Z$, then we would have $V(P) = V(P')$, hence $P \subseteq P'$ and $P' \subseteq P$, so P is unique. $\square$

2.8. *Definition.* A map $f : X \to Y$ of topological spaces is a *quasi-homeomorphism* if $U \mapsto f^{-1}(U)$ is a bijection from the open subsets of Y to the open subsets of X (or equivalently, from the closed subsets of Y to the closed subsets of X).

Any quasi-homeomorphism f is, in particular, continuous.

If f is an injective quasi-homeomorphism, then f is a homeomorphism of X onto its image $Y' = f(X) \subseteq Y$, and Y' has the property that $Z \cap Y'$ is dense in Z, for every closed $Z \subseteq Y$. Conversely, this property of a subspace $Y' \subseteq Y$ implies that the inclusion map is a quasi-homeomorphism.

A bijective quasi-homeomorphism is a homeomorphism.

If $f : X \to Y$ is a quasi-homeomorphism, then the functor f_* defines an equivalence of categories from presheaves on X to presheaves on Y, and likewise for sheaves. In other words, quasi-homeomorphic spaces effectively have identical sheaf theories.

2.9. Given any topological space X, let $\mathrm{Sob}(X)$ denote the set of irreducible closed subsets of X. For every closed $Y \subseteq X$, define

$$V(Y) = \{Z \in \mathrm{Sob}(X) \mid Z \subseteq Y\}.$$

The sets $V(Y)$ are the closed subsets of a topology on $\mathrm{Sob}(X)$, by virtue of the identities

$$V(\emptyset) = \emptyset, \quad V(X) = \mathrm{Sob}(X), \quad V(\bigcap_\alpha Y_\alpha) = \bigcap_\alpha V(Y_\alpha), \quad V(Y_1 \cup Y_2) = V(Y_1) \cup V(Y_2).$$

The first of these holds because an irreducible space is non-empty by definition. For the last, if $Z \in V(Y_1 \cup Y_2)$ then, since Z is irreducible, $Z \in V(Y_1)$ or $Z \in V(Y_2)$. Thus $V(Y_1 \cup Y_2) \subseteq V(Y_1) \cup V(Y_2)$, and the opposite containment is trivial. The remaining two identities above are trivial as well.

There is a canonical map

$$i \colon X \to \mathrm{Sob}(X)$$

defined by $i(x) = \overline{\{x\}}$. One checks immediately that $i^{-1}(V(Y)) = Y$ for every closed $Y \subseteq X$. The correspondence $Y \mapsto V(Y)$ from closed subsets of X to closed subsets of $\mathrm{Sob}(X)$ is surjective by definition. The identity $i^{-1}(V(Y)) = Y$ then implies that $V(-)$ and i^{-1} are inverse bijections between the closed subsets of X and $\mathrm{Sob}(X)$. In particular, we have the following.

2.10. *Proposition.* For any space X, the canonical map $i \colon X \to \mathrm{Sob}(X)$ is a quasi-homeomorphism.

2.11. *Proposition.* For any space X, $\mathrm{Sob}(X)$ is a sober space.

Proof. Since i is a quasi-homeomorphism, i^{-1} induces a bijection from irreducible closed subsets of $\mathrm{Sob}(X)$ to irreducible closed subsets of X, with inverse given by $V(-)$. Thus every irreducible closed subset of $\mathrm{Sob}(X)$ is $V(Z)$ for a unique irreducible closed $Z \subseteq X$. The smallest closed $Y \subseteq X$ such that $Z \in V(Y)$, that is, such that $Z \subseteq Y$, is clearly $Y = Z$, so $V(Z)$ is the smallest closed subset of $\mathrm{Sob}(X)$ containing Z, that is, $V(Z) = \overline{\{Z\}}$ in

$\mathrm{Sob}(X)$. For uniqueness, if $\overline{\{Z\}} = \overline{\{Z'\}}$, then $Z \in V(Z')$ and $Z' \in V(Z)$ imply $Z \subseteq Z' \subseteq Z$, so $Z = Z'$. $\square$

2.12. *Lemma.* If X is sober, then the canonical map $i\colon X \to \mathrm{Sob}(X)$ is a homeomorphism.

Proof. By Proposition 2.11, i is a quasi-homeomorphism. The assertion that i is a bijection is equivalent to the definition of the space X being sober. $\square$

2.13. Given a continuous map $f\colon X \to X'$, we define a map $\mathrm{Sob}(f)\colon \mathrm{Sob}(X) \to \mathrm{Sob}(X')$ by

$$\mathrm{Sob}(f)(Z) = \overline{f(Z)}.$$

This makes sense by Proposition 2.3, (i) and (ii). For $Y' \subseteq X'$ closed, we have $\mathrm{Sob}(f)(Z) \in V(Y') \Leftrightarrow f(Z) \subseteq Y' \Leftrightarrow Z \subseteq f^{-1}(Y')$. In other words, $\mathrm{Sob}(f)^{-1}(V(Y')) = V(f^{-1}(Y'))$. Hence $\mathrm{Sob}(f)$ is continuous. It is easy to see that $\mathrm{Sob}(g \circ f) = \mathrm{Sob}(g) \circ \mathrm{Sob}(f)$, so

$$\mathrm{Sob}(-)\colon \underline{\mathrm{Top}} \to (\text{sober spaces})$$

is a functor.

2.14. For any continuous map $f\colon X \to X'$, one checks immediately that the diagram

$$\begin{array}{ccc} \mathrm{Sob}(X) & \xrightarrow[\mathrm{Sob}(f)]{} & \mathrm{Sob}(X') \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

commutes. In other words, the canonical maps $i_X\colon X \to \mathrm{Sob}(X)$ give a functorial map from the identity functor $\mathrm{id}_{\underline{\mathrm{Top}}}$ to $j \circ \mathrm{Sob}$, where $j\colon (\text{sober spaces}) \to \underline{\mathrm{Top}}$ is the inclusion functor.

2.15. *Proposition.* The canonical map $i_X\colon X \to \mathrm{Sob}(X)$ has the universal property that every continuous map $f\colon X \to Y$, where Y is sober, factors uniquely through i_X. Equivalently, the functor $\mathrm{Sob}(-)$ is left adjoint to the inclusion $j\colon (\text{sober spaces}) \to \underline{\mathrm{Top}}$, with the functorial map in 2.14 giving the unit of the adjunction.

Proof. The equivalence of the two statements is easy and purely category-theoretic.

For the universal property, suppose $f\colon X \to Y$ is continuous and Y is sober. Then i_Y is a homeomorphism, by Lemma 2.12. Taking $X' = Y$ in diagram 2.14, we see that $f = g \circ i_X$, where $g = i_Y^{-1} \circ \mathrm{Sob}(f)$. Thus f factors through i_X.

To see that g is unique, we proceed as follows. If Z is an irreducible closed subset of X, then $\overline{i_X(Z)}$ is the smallest closed subset $V(Y)$ of $\mathrm{Sob}(X)$ such that $i_X(Z) \subseteq V(Y)$, or equivalently, such that $Z \subseteq i_X^{-1}(V(Y)) = Y$. But the smallest such Y is $Y = Z$, that is, we have $\overline{i_X(Z)} = V(Z)$. In the proof of Proposition 2.11, we saw that $V(Z) = \overline{\{Z\}}$. In other words, the maps $\mathrm{Sob}(i_X)$ and i_{Sob_X} from $\mathrm{Sob}(X)$ to $\mathrm{Sob}(\mathrm{Sob}(X))$ coincide. By Proposition 2.11 and Lemma 2.12, $i_{\mathrm{Sob}(X)}$ is a homeomorphism; thus $\mathrm{Sob}(i_X)$ is as well.

Now, $f = g \circ i_X$ implies $\mathrm{Sob}(f) = \mathrm{Sob}(g) \circ \mathrm{Sob}(i_X)$. Since $\mathrm{Sob}(i_X)$ is a homeomorphism, this condition determines $\mathrm{Sob}(g)$. But $g\colon \mathrm{Sob}(X) \to Y$ is a map between sober spaces. Lemma 2.12 and diagram 2.14 imply that for such a map, $\mathrm{Sob}(g)$ determines g. $\square$

Remark. The formula $g = i_Y^{-1} \circ \mathrm{Sob}(f)$ for the unique map g in the factorization $f = g \circ i_X$ means that for $Z \in \mathrm{Sob}(X)$, $g(Z)$ is the unique point $y \in Y$ such that $\overline{f(Z)} = \overline{\{y\}}$.

2.16. We conclude this section by explaining how sober spaces have a purely topological significance, apart from their appearance as the underlying spaces of schemes. We will not need this in what follows, so we leave some of the details as exercises for the reader.

Because the points of a sober space are in bijection with its irreducible closed subsets, any quasi-homeomorphism between sober spaces is bijective, and thus a homeomorphism. If $f\colon X \to Y$ is a quasi-homeomorphism between arbitrary topological spaces, it follows that $\mathrm{Sob}(f)$ is a homeomorphism. In other words, the functor $\mathrm{Sob}\colon \underline{\mathrm{Top}} \to$ (sober spaces) factors through a functor $S\colon Q^{-1}\,\underline{\mathrm{Top}} \to$ (sober spaces), where $Q^{-1}\,\underline{\mathrm{Top}}$ is the category obtained by formally inverting all quasi-homeomorphisms in $\underline{\mathrm{Top}}$.

Composing the canonical functor $\underline{\mathrm{Top}} \to Q^{-1}\,\underline{\mathrm{Top}}$ with the inclusion of sober spaces into $\underline{\mathrm{Top}}$ gives a functor $j\colon$ (sober spaces) $\to Q^{-1}\,\underline{\mathrm{Top}}$. Using Proposition 2.10 and Lemma 2.12, one can verify that S and j are inverse up to functorial isomorphism, giving an equivalence of categories

$$Q^{-1}\,\underline{\mathrm{Top}} \cong \text{(sober spaces)}.$$

In particular, the category $Q^{-1}\,\underline{\mathrm{Top}}$ (which is a priori a *large category*, that is, the class of morphisms between two objects need not be a set) is an ordinary category, for which the category of sober spaces serves as a concrete natural model.

3. Jacobson spaces

3.1. *Definition.* A topological space X is *Jacobson* if every closed subset $Z \subseteq X$ is equal to the closure of the set Z_{cl} of closed points in Z (that is, points p such that $\{p\}$ is closed; note that $Z_{\mathrm{cl}} = Z \cap X_{\mathrm{cl}}$, since Z is closed).

3.2. *Lemma.* The following conditions are equivalent:
(i) X is Jacobson.
(ii) The inclusion $X_{\mathrm{cl}} \hookrightarrow X$ is a quasi-homeomorphism.
(iii) Every non-empty locally closed subset of X contains a point of X_{cl}.

The proof is an easy exercise.

3.3. *Lemma.* (i) Every closed subset of a Jacobson space is Jacobson.
(ii) Every open subset of a Jacobson space is Jacobson.
(iii) If a space X has an open cover $X = \bigcup_\alpha X_\alpha$, where each X_α is Jacobson, then X is Jacobson.

Proof. Part (i) is immediate from the definition.

For (ii), let X be Jacobson, $U \subseteq X$ open, $W \subseteq U$ locally closed, $W \neq \emptyset$. Then W is also locally closed in X, hence contains a point $p \in X_{\mathrm{cl}}$. But p is then also a closed point of U. Using Lemma 3.2, (iii), this shows that U is Jacobson.

For (iii), let $W \subseteq X$ be locally closed and non-empty. Then $W \cap X_\alpha$ is locally closed and non-empty for some α, hence $W \cap X_\alpha$ contains a closed point p of X_α. We claim that p is a closed point of X (which is not obvious, since X_α need not be closed). We have $\{p\} = \overline{\{p\}} \cap X_\alpha$, so $\{p\}$ is locally closed. For every β such that $p \in X_\beta$, it follows that p is closed in X_β, since the locally closed subset $\{p\}$ must contain a closed point of X_β. Now suppose $q \in \overline{\{p\}}$. For some β, we have $q \in X_\beta$. Since $q \in \overline{\{p\}}$ and X_β is open, we also have

$p \in X_\beta$. But then q belongs to the closure of $\{p\}$ in X_β, so $q = p$. Again using Lemma 3.2, (iii), this shows that X is Jacobson. $\square$

3.4. For any commutative ring R, the closed points of $\operatorname{Spec}(R)$ are the maximal ideals $\mathfrak{m} \subseteq R$. The intersection of all maximal ideals $\mathfrak{m}$ containing a given ideal $I \subseteq R$ is called the *Jacobson radical* $\operatorname{rad}(I)$ of I. Clearly $\operatorname{rad}(I)$ is the unique radical ideal J such that $V(J) = \overline{Z_{\mathrm{cl}}}$, where $Z = V(I)$. Hence $\operatorname{Spec}(R)$ is Jacobson if and only if R has the property that $\operatorname{rad}(I) = \sqrt{I}$ for every ideal $I \subseteq R$. A ring R satisfying this condition is a *Jacobson ring*.

3.5. *Proposition.* Let X be a scheme. The following conditions are equivalent.

(i) X is Jacobson (meaning that its underlying topological space is Jacobson).

(ii) For every open affine $U = \operatorname{Spec}(R) \subseteq X$, the ring R is Jacobson.

(iii) There exists a covering $X = \bigcup_\alpha U_\alpha$ of X by open affines $U_\alpha = \operatorname{Spec}(R_\alpha)$ such that R_α is Jacobson.

Proof. Immediate from Lemma 3.3, (ii) and (iii).

3.6. Recall that, by definition, a topological space X is T_1 if every point of X is closed. If X is T_1, then the singleton sets $\{p\}$ are the minimal irreducible closed subsets of X, and are therefore the closed points of $\operatorname{Sob}(X)$. In other words, $i\colon X \to \operatorname{Sob}(X)$ is injective with image $i(X) = \operatorname{Sob}(X)_{\mathrm{cl}}$. By Proposition 2.10 and Lemma 3.2, (ii), it follows that $\operatorname{Sob}(X)$ is Jacobson, and i is a homeomorphism of X onto $\operatorname{Sob}(X)_{\mathrm{cl}}$. If $f\colon X \to X'$ is a continuous map of T_1 spaces, we also see that $\operatorname{Sob}(f)$ carries $\operatorname{Sob}(X)_{\mathrm{cl}}$ into $\operatorname{Sob}(X')_{\mathrm{cl}}$.

Conversely, if Y is a Jacobson sober space, then $j\colon Y_{\mathrm{cl}} \hookrightarrow Y$ is a quasi-homeomorphism, and therefore the induced map $\operatorname{Sob}(Y_{\mathrm{cl}}) \to Y$ is a homeomorphism.

Let $\underline{\text{JacSob}}$ denote the category of Jacobson sober spaces and continuous maps $f\colon Y \to Y'$ such that $f(Y_{\mathrm{cl}}) \subseteq Y'_{\mathrm{cl}}$. The preceding observations prove the following result.

Proposition. The functor $\operatorname{Sob}(-)$ restricts to an equivalence of categories

$$\operatorname{Sob}(-) : (T_1 \text{ spaces}) \to \underline{\text{JacSob}}$$

with inverse $Y \mapsto Y_{\mathrm{cl}}$. $\square$

4. Finite morphisms

4.1. *Definition.* A morphism of affine schemes $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is *finite* if the corresponding ring homomorphism $R \to S$ makes S an R-algebra finitely generated as an R-module.

We omit the definition of a finite morphism of general schemes, as we will only need the affine case here.

4.2. Let S be an R-algebra. An element $s \in S$ is *integral* over R if there is a monic polynomial $f(x) \in R[x]$ such that $f(s) = 0$. If S is generated as an R-algebra by an integral element s, then we can use the relation $f(s) = 0$ to reduce any power s^n, where $n \geq \deg(f)$, to an R-linear combination of smaller powers of s. It follows that S is generated as an R-module by the finite set of powers s^n for $n < \deg(f)$.

This generalizes easily to show that S is finitely generated as an R-module if it is generated as an R-algebra by a finite set of elements integral over R, but here we will only need the case of a single integral element.

4.3. *Proposition* (Nakayama's lemma). Let R be a local ring with maximal ideal P, and let M be a finitely generated R-module. If $M/PM = 0$, then $M = 0$.

For completeness, we give the standard proof:

Proof. Suppose $x_1, \ldots, x_n$ generate M. By hypothesis, $PM = M$, so each x_i belongs to PM. Hence we can write $x_i = \sum_j a_{ij} x_j$, with all $a_{ij} \in P$. Let A be the $n \times n$ matrix with entries a_{ij}. By construction, we have $(I - A)v = 0$, where $v \in M^n$ is the vector with entries x_j. The determinant $\det(I - A) \in R$ is congruent to 1 modulo P, hence it is a unit in R, so the matrix $I - A$ is invertible. Hence all the x_j are zero, that is, $M = 0$. □

Nakayama's lemma has the following important geometric consequence.

4.4. *Proposition.* Let $f\colon X = \operatorname{Spec}(S) \to \operatorname{Spec}(R) = Y$ be a finite morphism of affine schemes. Then $f(X)$ is closed in Y.

Proof. The closure $\overline{f(X)}$ is $V(I)$, where I is the kernel of the ring homomorphism $R \to S$ corresponding to f. Replacing R with R/I, we can assume that $R \to S$ is injective, and (hence) $\overline{f(X)} = Y$. In this case we are to prove that f is surjective.

Let P be a point of $\operatorname{Spec}(R)$. The scheme-theoretic fiber $f^{-1}(P)$, whose underlying space is homeomorphic to the topological fiber $f^{-1}(P)$, is $\operatorname{Spec}(\mathbf{k}_P \otimes_R S) = \operatorname{Spec}(S_P/PS_P)$. Since $R \to S$ is injective, so is the localization $R_P \to S_P$. In particular, $S_P \neq 0$. Since f is finite, S_P is a finitely generated R_P-module. By Nakayama's lemma, it follows that $S_P/PS_P \neq 0$, that is, $f^{-1}(P)$ is non-empty. □

The same proposition for a finite morphism of arbitrary schemes follows as a corollary to the affine case, but we will not need it here.

4.5. *Proposition.* Let S be an integral domain and $R \subseteq S$ a subring such that S is a finitely generated R-module. Let $f\colon X = \operatorname{Spec}(S) \to \operatorname{Spec}(R) = Y$ be the corresponding finite morphism (which is surjective by Proposition 4.4). Then, for every nonempty open subset $U \subseteq X$, $f(U)$ contains a non-empty open subset of Y.

Proof. We can always replace U with a smaller open subset and so assume that $U = X_a$ for some non-zero $a \in S$.

Since R is a subring of S, R is a domain. Let K be the fraction field of R. Then $K \otimes_R S = KS$ is a subring of the fraction field of S, hence also a domain, and KS is a finitely generated K-module, that is, a finite dimensional vector space over K. This implies that KS is a field: if $x \in KS$ is non-zero, then multiplication by x is an injective K-linear map $\mu_x\colon KS \to KS$. Since KS is finite dimensional, μ_x is bijective, and $y = \mu_x^{-1}(1)$ is then a multiplicative inverse of x.

Let b/h be the inverse of a in KS, where $b \in S$ and $h \in R$. Then $h = ab$ in S, so S_h contains S_a, which means that $f^{-1}(Y_h) \subseteq U$. Since f is surjective, this implies that $Y_h \subseteq f(U)$. □

We remark that the proof actually shows that U contains the preimage $f^{-1}(W)$ of some non-empty open $W \subseteq Y$. Equivalently, for every proper closed subset $Z \subsetneq X$, $f(Z)$ (which is closed, by Proposition 4.4) is a proper subset of Y.

5. Geometry of the affine line

5.1. For any commutative ring R, the scheme $\mathbb{A}^1_R = \operatorname{Spec}(R[x])$, considered as a scheme over R via the morphism $\pi\colon \mathbb{A}^1_R \to \operatorname{Spec}(R)$ coming from the inclusion $R \hookrightarrow R[x]$, is the *affine line over* $\operatorname{Spec}(R)$. In this section we investigate the geometry of the morphism π. As we shall see later, the results are key to understanding more general morphisms locally of finite type.

5.2. We first consider the simplest case, when $R = k$ is a field. Then $k[x]$ is a principal ideal domain, whose prime ideals are (0) and ideals of the form $(f(x))$, where $f(x)$ is an irreducible polynomial. The ideals $(f(x))$ are maximal, that is, they are closed points of $\mathbb{A}^1_k$. The ideal (0) gives the generic point $Q \in \mathbb{A}^1_k$, whose closure is the whole affine line. Since a general non-zero ideal has the form $(g(x))$, and $g(x)$ has a finite set of irreducible factors, we see that the closed subsets of $\mathbb{A}^1_k$ are the whole line and all finite sets of closed points.

The set of distinct monic irreducible polynomials over k is always infinite. This is obvious if k is an infinite field, but also true if k is finite—in fact, there are irreducible polynomials of every degree $d > 0$. The set of all closed points of $\mathbb{A}^1_k$ is therefore not closed, hence its closure is the whole line. From this description we see in particular that $\mathbb{A}^1_k$ is Jacobson.

In general, the fibers of the morphism $\pi\colon \mathbb{A}^1_R \to \operatorname{Spec}(R)$ are of the form $\mathbb{A}^1_k$, so we have a precise picture of their topology.

Beyond the fiber-wise picture, we need some information on how the fibers fit together. Note that if R is a domain, then so is $R[x]$, hence $\mathbb{A}^1_Z$ is irreducible for any irreducible affine scheme Z (and consequently also for any irreducible scheme Z).

5.3. *Proposition.* In terms of the morphism $\pi\colon \mathbb{A}^1_R \to \operatorname{Spec}(R)$, the irreducible closed subsets $Z \subseteq \mathbb{A}^1_R$ are of two types:

(i) the preimage $Z = \pi^{-1}(Y) \cong \mathbb{A}^1_Y$ of an irreducible closed subset $Y \subseteq \operatorname{Spec}(R)$; or

(ii) the closure $Z = \overline{W}$ of an irreducible locally closed subset W such that π restricts to a finite morphism $W \to Y$, where $Y = \pi(W)$ is an irreducible locally closed subset of $\operatorname{Spec}(R)$.

Proof. Let $Z = V(P)$, where $P \subseteq R[x]$ is a prime ideal. The closure of $\pi(Z)$ is then $V(Q)$, where $Q = P \cap R$, and either conclusion (i) or (ii) follows from the same conclusion with R replaced by R/Q. Thus we can assume without loss of generality that R is a domain, and $P \cap R = 0$. If $P = 0$, we have conclusion (i). Otherwise, let K be the fraction field of R. Then $K[x]$ is the localization $S^{-1}R[x]$, where the multiplicative set $S = R^\times$ is disjoint from P, so $PK[x]$ is a non-zero prime ideal of $K[x]$, of the form $(f(x))$, where $f(x)$ is an irreducible polynomial over K. Multiplying by an element of $R^\times$ to clear denominators in the coefficients, we can assume that $f(x) \in R[x]$. Note that $PK[x] \cap R[x] = S^{-1}P \cap R[x] = P$ (geometrically, this says that the unique point of Z belonging to the fiber of $\mathbb{A}^1_R$ over the generic point of $\operatorname{Spec}(R)$ is the generic point of Z). In particular, $f(x) \in P$.

Let $a \in R^\times$ be the coefficient of the highest degree term in $f(x)$. Dividing by a gives a monic polynomial $g(x)$ such that $(g(x)) = (f(x))$ in $R_a[x]$. Then $Y' = \operatorname{Spec}(R_a)$ is an open affine subscheme of $\operatorname{Spec}(R)$, and $W = \pi^{-1}(Y') \cap Z = Z_a = \operatorname{Spec}(R_a[x]/PR_a[x])$ is an irreducible component of $V(g(x)) \subseteq \mathbb{A}^1_{Y'}$, which is finite over Y', since g is monic (§4.2). By Proposition 4.4, $Y = \pi(W)$ is closed in Y', thus locally closed in $\mathrm{Spec}(R)$. Since $W \to Y$ is again finite, this yields conclusion (ii). $\square$

Note that in the special case where $R = k$ is a field, the proposition restates the fact that the irreducible closed subsets of $\mathbb{A}^1_k$ are the whole line and the closed points.

5.4. *Proposition.* The morphism $\pi\colon \mathbb{A}^1_R \to \mathrm{Spec}(R)$ has the property that if Z is a non-empty locally closed subset of $\mathbb{A}^1_R$, then $\pi(Z)$ contains a non-empty locally closed subset of $\mathrm{Spec}(R)$.

Proof. If the conclusion of the proposition holds for any non-empty locally closed subset $Z' \subseteq Z$, then it also holds for Z. Hence we are free to shrink Z, as long as we keep it locally closed and non-empty. In particular, we can assume without loss of generality that Z is irreducible. We can also assume, just as we did in the proof of Proposition 5.3, that R is a domain and that $P \cap R = 0$, where the prime ideal $P \subseteq R[x]$ is the generic point of Z, that is, $\overline{Z} = V(P)$. Geometrically, this means that $\mathrm{Spec}\, R$ is irreducible (and reduced) and that π maps the the generic point P of Z to the generic point Q of $\mathrm{Spec}(R)$.

Suppose that the closure $\overline{Z}$ is of type (i) in Proposition 5.3. Then Z, which is open in $\overline{Z}$, contains a non-empty open subset of the fiber $\pi^{-1}(Q)$. Since this fiber is an affine line over the fraction field K of R, Z contains some point z (in fact, infinitely many such points) closed in $\pi^{-1}(Q)$. We can then shrink Z to $Z \cap \overline{\{z\}}$, and so assume without loss of generality that $\overline{Z}$ meets $\pi^{-1}(Q)$ but does not contain its generic point, hence belongs to type (ii) in Proposition 5.3.

Now let W and Y be as in Proposition 5.3 (ii) for $\overline{Z}$. We can take W and Y to be reduced and affine, as well as irreducible, so the finite surjective morphism $W \to Y$ corresponds to an injective homomorphism between integral domains. Shrinking Z further, we can assume that Z is an open subset of W. Then Proposition 4.5 shows that $\pi(Z)$ contains a non-empty open subset of Y, as desired. $\square$

6. Morphisms locally of finite type

6.1. *Definition.* A morphism of schemes $f\colon X \to Y$ is *locally of finite type* if for every point $x \in X$, there exist open affine neighborhoods $x \in U = \mathrm{Spec}(B) \subseteq X$ and $f(y) \in V = \mathrm{Spec}(A) \subseteq Y$ such that $f(U) \subseteq V$, and the ring homomorphism $A \to B$ corresponding to the morphism $(f|U)\colon U \to V$ makes B a finitely generated A-algebra.

6.2. Suppose $U = \mathrm{Spec}(B)$ and $V = \mathrm{Spec}(A)$ satisfy the conditions in Definition 6.1. We can replace U with any smaller basic open neighborhood $U_g = \mathrm{Spec}\, B[g^{-1}]$ of x, since $B[g^{-1}]$ is a finitely generated B-algebra, hence a finitely generated A-algebra.

We can also replace V with any smaller open affine $V' = \mathrm{Spec}(A')$ such that $f(U) \subseteq V' \subseteq V$. If we do this, the morphisms $U \to V' \hookrightarrow V$ correspond to ring homomorphism $A \to A' \to B$, and any finite set of generators for B as an A-algebra also generates B as an A'-algebra. Note that we can take V' to be an arbitrary affine neighborhood of $f(x)$ in V if we also replace U with a basic open U_g such that $U_g \subseteq f^{-1}(V')$.

It follows that the definition is local on both X and Y. More precisely: (i) if $f\colon X \to Y$ is locally of finite type, then so is $(f|U)\colon U \to V$, for all open subschemes $U \subseteq X$ and $V \subseteq Y$ such that $f(U) \subseteq V$; (ii) if Y has a covering by open subschemes V_α such that $f^{-1}(V_\alpha) \to V_\alpha$

is locally of finite type for all α, then f is locally of finite type; and (iii) if X has a covering by open subschemes U_α such that $(f|U_\alpha)\colon U_\alpha \to Y$ is locally of finite type for all α, then f is locally of finite type.

In particular, since the identity morphism on any scheme is obviously locally of finite type, it follows that every open embedding is locally of finite type. It is also easy to see that every closed embedding is locally of finite type.

6.3. *Proposition.* If $f\colon X \to Y$ and $g\colon Y \to Z$ are locally of finite type, then so is $g \circ f$.

Proof. Given $x \in X$, let $y = f(x)$ and $z = g(y)$. We can find affine open neighborhoods $y \in V = \operatorname{Spec}(B) \subseteq Y$ and $z \in W = \operatorname{Spec}(A) \subseteq Z$ such that $g(V) \subseteq W$ and B is a finitely generated A-algebra. Using the observations in §6.2, we can find a basic open neighborhood $V_h = \operatorname{Spec}(B[h^{-1}]) \subseteq V$ of y and an affine neighborhood $U = \operatorname{Spec}(C) \subseteq X$ of x such that $f(U) \subseteq V_h$ and C is a finitely generated $B[h^{-1}]$-algebra. We now have $A \to B \to B[h^{-1}] \to C$ with each ring a finitely generated algebra over the previous one, so C is a finitely generated A-algebra. □

6.4. *Proposition.* Given $f\colon X \to Y$ and $g\colon Y \to Z$, if $g \circ f$ is locally of finite type, then so is f.

Proof. Given $x \in X$, let $y = f(x)$ and $z = g(y)$. We can find $x \in U = \operatorname{Spec}(C) \subseteq X$ and $z \in W = \operatorname{Spec}(A) \subseteq Z$ such that $(g \circ f)(U) \subseteq W$ and C is a finitely generated A-algebra. Choose any open affine neighborhood $V = \operatorname{Spec}(B) \subseteq Y$ of y such that $g(V) \subseteq W$. Then we can find a basic open subset $U_h \subseteq U$ such that $x \in U_h \subseteq f^{-1}(V)$. The morphisms $U_h \to V \to W$ correspond to ring homomorphisms $A \to B \to C[h^{-1}]$. Since C is a finitely generated A-algebra, so is $C[h^{-1}]$. Hence $C[h^{-1}]$ is also a finitely generated B-algebra. □

This proposition implies, in particular, that every S-morphism between schemes locally of finite type over S is locally of finite type.

6.5. *Proposition.* If $f\colon X \to Y$ is locally of finite type, then for *all* open affines $U = \operatorname{Spec}(B) \subseteq X$ and $V = \operatorname{Spec}(A) \subseteq Y$ such that $f(U) \subseteq V$, B is a finitely generated A-algebra. In particular, if $\varphi\colon A \to B$ is a ring homomorphism for which the corresponding morphism of affine schemes $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is locally of finite type, then B is finitely generated as an A-algebra.

Proof. By §6.2, the morphism $(f|U)\colon U \to V$ is locally of finite type; moreover, we can find pairs $U_g = \operatorname{Spec}(B[g^{-1}]) \subseteq U$ and $V_h = \operatorname{Spec}(A[h^{-1}]) \subseteq V$ such that $f(U_g) \subseteq V_h$ and $B[g^{-1}]$ is a finitely generated $A[h^{-1}]$-algebra, hence also a finitely generated A-algebra, with the open sets U_g in these pairs covering U. Since U is quasi-compact, we can take the covering $U = \bigcup_i U_{g_i}$ to be finite.

Thus we now have an A-algebra B and elements $g_1, \ldots, g_n \in B$ such that $B[g_i^{-1}]$ is a finitely generated A-algebra for each i, and the g_i generate the unit ideal in B. We are to show that B is a finitely generated A-algebra. Let

$$a_1 g_1 + \cdots + a_n g_n = 1.$$

For each $i = 1, \ldots, n$, we can take $B[g_i^{-1}]$ to be generated as an A-algebra by g_i^{-1} and the images in $B[g_i^{-1}]$ of finitely many elements $x_{ij} \in B$. Let B' be the A-subalgebra of B

generated by all the elements a_i, g_i, and x_{ij}. Let $U' = \operatorname{Spec}(B')$ and consider the sheaf $\widetilde{B}$ on U' associated to the B'-module B.

The inclusion $B' \hookrightarrow B$ induces a sheaf homomorphism $\mathcal{O}_{U'} \hookrightarrow \widetilde{B}$. We have $B'[g_i^{-1}] = B[g_i^{-1}]$, since B' contains all the x_{ij}. Hence the homomorphism $\mathcal{O}_{U'} \hookrightarrow \widetilde{B}$ restricts to an isomorphism on U'_{g_i}. The g_i generate the unit ideal in B', with the same coefficients a_i as in B, so the open subsets U'_{g_i} cover U'. It follows that $\mathcal{O}_{U'} \hookrightarrow \widetilde{B}$ is an isomorphism, hence so is $B' \hookrightarrow B$, that is, $B' = B$. But B' is a finitely generated A-algebra by construction. $\square$

7. Main theorem on Jacobson schemes

7.1. *Theorem.* Let $f\colon X \to Y$ be a morphism of schemes locally of finite type, where Y is Jacobson. Then:

(i) X is Jacobson.

(ii) $f(X_{\mathrm{cl}}) \subseteq Y_{\mathrm{cl}}$.

(iii) For every $x \in X_{\mathrm{cl}}$, with $y = f(x)$, the homomorphism of residue fields $\mathbf{k}_y \hookrightarrow \mathbf{k}_x$ induced by f is a finite algebraic extension.

7.2. The remainder of this section contains the proof of Theorem 7.1. The first step is to observe that the local nature of the hypothesis and the conclusions allows us to reduce to the case where X and Y are affine. This requires a bit of care regarding conclusion (ii), because it is not generally true that a closed point p of an open subset W of Y is a closed point of Y. However, this does hold given that Y is Jacobson, since the set $\overline{\{p\}} \cap W = \{p\}$ is locally closed, hence contains a closed point of Y.

7.3. We now assume that $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$. By Proposition 6.5, B is a finitely generated A-algebra, that is, $B = A[x_1, \ldots, x_n]/I$. We can factor the morphism $X \to Y$ into a chain of morphisms

$$X \to Y_n \to \cdots \to Y_1 \to Y_0 = Y,$$

where $Y_i = \operatorname{Spec}(A[x_1, \ldots, x_i])$. Assuming that Theorem 7.1 holds for each step in the chain, we can conclude first that all the Y_i and X are Jacobson, and then that the other conclusions of the theorem also hold for each morphism $Y_i \to Y$ and for $X \to Y$.

The conclusions of Theorem 7.1 are trivial for the morphism $X \to Y_n$, which is a closed embedding. We are left to verify Theorem 7.1 for the morphisms $Y_{i+1} \to Y_i$, that is, for morphisms of the form $\pi\colon \mathbb{A}^1(R) \to \operatorname{Spec}(R)$.

7.4. For the remaining case where $Y = \operatorname{Spec}(R)$ and $X = \mathbb{A}^1_R$, we use Proposition 5.4.

Let P be a closed point of $X = \mathbb{A}^1_R$. Since $\{P\}$ is closed, the singleton set $\{f(P)\}$ is locally closed, by Proposition 5.4. Assuming Y to be Jacobson, this implies that $f(P)$ is a closed point of Y, giving conclusion (ii) of Theorem 7.1.

The closed point P is a maximal ideal of $R[x]$, and we have just shown that $Q = f(P) = R \cap P$ is a maximal ideal of R. We have $\mathbf{k}_P = R[x]/P$ and $\mathbf{k}_Q = R/Q$, so the field $\mathbf{k}_P$ is a quotient of $\mathbf{k}_Q[x]$. Hence $\mathbf{k}_P = \mathbf{k}_Q[x]/(f(x))$, where $f(x)$ is an irreducible polynomial over $\mathbf{k}_Q$, and thus $\mathbf{k}_P$ is a finite algebraic extension of $\mathbf{k}_Q$. This gives conclusion (iii) of Theorem 7.1.

Finally, to show that X is Jacobson, let $Z \subseteq X$ be any non-empty locally closed subset. We want to show that Z contains a closed point p of X.

Since Y is Jacobson, Proposition 5.4 implies that $f(Z)$ contains a closed point $y \in Y$. Then $Z \cap f^{-1}(\{y\})$ is a non-empty locally closed subset of the fiber $f^{-1}(\{y\})$. This fiber is homeomorphic to the affine line $\mathbb{A}^1_k = \operatorname{Spec}(k[x])$, where $k = \mathbf{k}_y$. As we observed in §5.1, $\mathbb{A}^1_k$ is Jacobson. Hence $Z \cap f^{-1}(\{y\})$ contains a closed point p of $f^{-1}(\{y\})$. Since $f^{-1}(\{y\})$ is closed, p is a closed point of X.

8. Interpretation of classical varieties as schemes

8.1. We now have the tools needed to establish the equivalence between classical varieties and reduced algebraic schemes over an algebraically closed field k, as promised in the introduction. We start by describing in detail the constructions that give each direction of the equivalence.

8.2. Let X be a classical algebraic variety over an algebraically closed field k, as defined in §1.8. Let $\operatorname{Sob}(X)$ be the soberization of the underlying space of X, with $i\colon X \to \operatorname{Sob}(X)$ the canonical map. By Proposition 2.10, i is a quasi-homeomorphism, and therefore, as in §2.8, the direct image functor i_* is an equivalence between sheaves on X and sheaves on $\operatorname{Sob}(X)$. This gives $\operatorname{Sob}(X)$ the structure of a ringed space $(\operatorname{Sob}(X), i_*\mathcal{O}_X)$.

We will prove below that $(\operatorname{Sob}(X), i_*\mathcal{O}_X)$ is a reduced algebraic scheme over k. This construction will give the desired equivalence in the direction from varieties to schemes.

We remark that, although the equivalence means that $i_*\mathcal{O}_X$ is in some sense the 'same' as $\mathcal{O}_X$, we can no longer identify it with a sheaf of functions on $\operatorname{Sob}(X)$, because its sections do not take values in k at non-closed points $p \in \operatorname{Sob}(X)$, which do not belong to $i(X)$.

8.3. To construct the equivalence in the opposite direction, let k be an algebraically closed field, and let Y be a reduced scheme algebraic (that is, locally of finite type) over k. By Theorem 7.1 (i) for the defining morphism $Y \to \operatorname{Spec}(k)$ of Y as a k-scheme, Y is Jacobson. By Theorem 7.1 (iii), the residue field at every closed point $p \in Y_{\mathrm{cl}}$ is isomorphic to k, via a canonical isomorphism $k \to \mathbf{k}_p$ determined by the given morphism $Y \to \operatorname{Spec}(k)$. For every open subset $U \subseteq Y$, the evaluation maps $\mathcal{O}_Y(U) \to \mathbf{k}_p = k$ for $p \in U \cap Y_{\mathrm{cl}}$ associate a k-valued function on $U \cap Y_{\mathrm{cl}}$ to each section $f \in \mathcal{O}_Y(U)$. This gives a homomorphism of sheaves of rings $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{F}^k_X$, where $X = Y_{\mathrm{cl}}$ and $i\colon X \to Y$ is the inclusion map. We define $\mathcal{O}_X \subseteq \mathcal{F}^k_X$ to be the sheaf of k-valued functions on X such that $i_*\mathcal{O}_X$ is the sheaf image of the homomorphism $i^\flat$.

We will prove below that the space X, with its sheaf of k-valued functions $\mathcal{O}_X$, is a classical algebraic variety over k. This construction will give the desired equivalence from schemes to varieties.

8.4. The next step is to verify that each of the constructions in §8.2 and §8.3 produces a space of the required type. Since general classical varieties (resp. schemes) are defined as ringed spaces having an open cover by affine varieties (resp. schemes), and the two constructions are local in nature, it suffices to verify the affine case.

In this case, we would also like to show more explicitly that the affine variety X with coordinate ring R corresponds to the scheme $Y = \operatorname{Spec}(R)$.

Theorem. Let k be an algebraically closed field.

(a) If X is a classical affine variety over k with coordinate ring $R = \mathcal{O}(X)$, then the ringed space $(\mathrm{Sob}(X), i_*\mathcal{O}_X)$ constructed in §8.2 is isomorphic to the affine scheme $Y = \mathrm{Spec}(R)$, and Y is reduced and algebraic over k.

(b) If $Y = \mathrm{Spec}(R)$ is a reduced algebraic affine scheme over k, then the space $X = Y_{\mathrm{cl}}$, with the sheaf of k-valued functions $\mathcal{O}_X$ defined in §8.3, is a classical affine algebraic variety over k with $\mathcal{O}(X) = R$.

8.5. For the proof of Theorem 8.4, we need the following lemma.

Lemma. Let k be an algebraically closed field. Every reduced, finitely generated k-algebra R is isomorphic to the coordinate ring $\mathcal{O}(X)$ of a classical affine variety X over k.

Proof. Choosing generators for R, we have a presentation $R = k[x_1, \dots, x_n]/I$, where $\sqrt{I} = I$, and we are to show that $I = \mathcal{I}(X)$, where X is the closed subvariety variety $V(I) \subseteq k^n$.

Theorem 7.1 (i) applied to $\mathrm{Spec}(k[\mathbf{x}]) \to \mathrm{Spec}(k)$ implies that the polynomial ring $k[\mathbf{x}]$ is Jacobson. By Theorem 7.1 (iii), since k is algebraically closed, every maximal ideal $\mathfrak{m} \subseteq k[\mathbf{x}]$ has $k[\mathbf{x}]/\mathfrak{m} = k$. In other words, every maximal ideal is the kernel $\mathfrak{m}_a$ of the k-algebra homomorphism $\mathrm{ev}_a\colon k[\mathbf{x}] \to k$ for some point $a \in k^n$, as in §1.7.

The maximal ideals containing I correspond to points $a \in X = V(I)$. Since R is Jacobson, their intersection is $\sqrt{I} = I$. This shows that $I = \mathcal{I}(X)$. □

Remark. This lemma, or the equivalent statement that $\mathcal{I}(V(I)) = \sqrt{I}$ for every ideal $I \subseteq k[\mathbf{x}]$, is Hilbert's Nullstellensatz. From the proof we see that the Nullstellensatz is a corollary to Theorem 7.1.

8.6. We now prove Theorem 8.4, part (b). Let $Y = \mathrm{Spec}(R)$, where R is a reduced finitely generated k-algebra and k is an algebraically closed field. By Lemma 8.5, we have $R = \mathcal{O}(X)$ for a classical affine variety X. In §1.7 we saw how to reconstruct X from R. There we did not assume that every maximal ideal $\mathfrak{m} \subseteq R$ has residue field $R/\mathfrak{m} = k$, but we now know that this is true, thanks to Theorem 7.1, so the construction in §1.7 identifies X with the space Y_{cl}. Comparing this with the construction in §8.3, it is clear that the space constructed in §8.3 is isomorphic to the variety X with its sheaf of regular functions $\mathcal{O}_X$.

8.7. For Theorem 8.4, part (a), let $X \subseteq k^n$ be a classical affine variety with coordinate ring $R = \mathcal{O}(X) = k[\mathbf{x}]/\mathcal{I}(X)$. The ring R is a finitely generated k-algebra by definition. It is reduced because R is a ring of k-valued functions on X. Hence $Y = \mathrm{Spec}(R)$ is a reduced algebraic scheme over k.

By part (b), we can identify X with Y_{cl} and $\mathcal{O}_X$ with the sheaf of functions such that $i_*\mathcal{O}_X$ is the sheaf image of $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{F}^k_X$. Since Y is sober and Jacobson, Y is canonically homeomorphic to $\mathrm{Sob}(X) = \mathrm{Sob}(Y_{\mathrm{cl}})$, by Proposition 3.6.

The sheaf homomorphism $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{O}_X$ is surjective by construction. To prove that it is injective, we need to show for every open subset $U \in Y$ that if $f \in \mathcal{O}_Y(U)$ evaluates to the function $i^\flat(f) = 0$ on $X \cap U$, then $f = 0$. It suffices to verify this in the case where $U = Y_h$ and $\mathcal{O}_Y(U) = R_h$. As in §7.2, we have $X \cap U = Y_{\mathrm{cl}} \cap U = U_{\mathrm{cl}}$ because Y is Jacobson. Thus $i^\flat(f) = 0$ means that f belongs to every maximal ideal of R_h. Now R_h is again Jacobson, so the intersection of its maximal ideals is the radical $\sqrt{0}$ (see §3.4). Since R_h is also reduced, $\sqrt{0} = 0$.

This shows that $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{O}_X$ is an isomorphism, making the ringed space $(\mathrm{Sob}(X), i_*\mathcal{O}_X)$ isomorphic to $(Y, \mathcal{O}_Y)$.

8.8. Theorem 8.4 implies that the constructions in §8.2 and §8.3 produce spaces of the right type, and also that for affine varieties and schemes, they are inverse to one another other, up to canonical isomorphism. Passing to affine coverings, it follows that the the two constructions are also inverse to one another in general.

In other words, given an algebraically closed field k, the two constructions give an equivalence between objects in the category of classical varieties over k and objects in the category of reduced algebraic schemes over k. What remains to be shown is that this extends to an equivalence of categories.

Theorem. Let k be an algebraically closed field. The constructions in §8.2 and §8.3 give a functorial equivalence between the categories of classical varieties and reduced affine schemes over k. Specifically, morphisms $f\colon X \to X'$ between classical varieties are in functorial one-to-one correspondence with k-morphisms $g\colon Y \to Y'$ between the schemes associated to them by the soberization construction in §8.2.

8.9. Before proving Theorem 8.8, we take care of some unfinished business on classical affine varieties.

The first point is to prove that if X is a classical affine variety with coordinate ring $R = \mathcal{O}(X)$, then the canonical homomorphism $R \to \mathcal{O}_X(X)$ is an isomorphism. In other words, every global regular function on X is a polynomial in the coordinates.

This now follows from Theorem 8.4 and the corresponding theorem $\mathcal{O}_Y(Y) = R$ for the affine scheme $Y = \mathrm{Spec}(R)$. To be precise, we have $Y \cong (\mathrm{Sob}(X), i_*\mathcal{O}_X)$, hence $\mathcal{O}_X(X) = \mathcal{O}_Y(Y) = R$. More generally, identifying X with Y_{cl}, we have $X_f = X \cap Y_f$ for every $f \in R$, hence the canonical homomorphism $\rho_f\colon R_f \to \mathcal{O}_X(X_f)$ is an isomorphism, as mentioned in §1.6.

The second point is to justify the assertion in §1.10, that if X is a classical affine variety, and $f \in R = \mathcal{O}(X)$ is a polynomial function on X, then the open subspace X_f, with the sheaf of functions $\mathcal{O}_X|X_f$, is isomorphic to a classical affine variety.

By Theorem 8.4, we have $X = Y_{\mathrm{cl}}$, where $Y = \mathrm{Spec}(R)$, with $\mathcal{O}_X$ the sheaf of functions constructed in §8.3. The open affine subscheme $Y_f = \mathrm{Spec}(R_f)$ also corresponds to a classical affine variety X'. The underlying space of X' is $(Y_f)_{\mathrm{cl}} = Y_{\mathrm{cl}} \cap Y_f = X_f$, and its sheaf of functions, constructed from $\mathcal{O}_{Y_f} = \mathcal{O}_Y|Y_f$, coincides with $\mathcal{O}_X|X_f$. Hence $(X_f, \mathcal{O}_X|X_f)$ is isomorphic to the classical affine variety X' with coordinate ring R_f.

8.10. We now prove Theorem 8.8.

As we observed in §1.10, it follows from §8.9 that on every classical variety X, the open subsets U isomorphic to classical affine varieties form a base of the topology. To give a morphism $f\colon X \to X'$ between classical varieties, it is therefore equivalent to give a system of compatible morphisms $f_U\colon U \to W$ from open affine subvarieties U covering X to open affine subvarieties $W \subseteq X'$. Morphisms $g\colon Y \to Y'$ between schemes, of course, have a similar local description. This reduces the proof of Theorem 8.8 to the affine case.

As noted in §1.9, the identity $\mathcal{O}(X) = \mathcal{O}_X(X)$ proven in §8.9 implies that if X and X' are classical affine varieties over k with coordinate rings $R = \mathcal{O}(X)$ and $R' = \mathcal{O}(X')$, then morphisms $f\colon X \to X'$ correspond bijectively with k-algebra homomorphisms $\varphi\colon R' \to$

R. This correspondence is clearly functorial, being given by a special case of the functor $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$ from the category of all ringed spaces to the opposite of the category of rings.

We also have a functorial bijection from k-algebra homomorphisms $\varphi\colon R' \to R$ to k-morphisms $g\colon Y \to Y'$ between the affine k-schemes $Y = \operatorname{Spec}(R)$ and $Y' = \operatorname{Spec}(R')$.

Hence, in the affine case, we have functorial equivalences between three categories: (i) the category of classical affine varieties X over k; (ii) the opposite of the category of reduced, finitely generated k-algebras R; and (iii) the category of reduced algebraic affine schemes Y over k. The functorial equivalence between (i) and (iii) then extends to the non-affine case as previously explained. $\square$

9. Variations on the theme

9.1. Some aspects of the theory developed above are relevant to schemes locally of finite type over any Jacobson base scheme S. This includes, for example, algebraic schemes over any field k, and also schemes locally of finite type over $S = \operatorname{Spec}(\mathbb{Z})$.

9.2. Fix a Jacobson scheme S. By Theorem 7.1 (i) and Proposition 2.7, every scheme Y locally of finite type over S is a Jacobson sober space. By Proposition 3.6, Y is the soberization of its subspace of closed points Y_{cl}, and the inclusion $i\colon Y_{\mathrm{cl}} \hookrightarrow Y$ is a quasi-homeomorphism, inducing an equivalence between sheaves on Y_{cl} and sheaves on Y. Hence $X = Y_{\mathrm{cl}}$, equipped with the sheaf of rings $\mathcal{O}_X = i^{-1}\mathcal{O}_Y$, is a T_1 locally ringed space characterized up to unique isomorphism by the identification $Y \cong (\operatorname{Sob}(X), i_*\mathcal{O}_X)$.

By Theorem 7.1 (iii), the residue field $\mathbf{k}_x$ at each point $x \in X$ is finite algebraic over $\mathbf{k}_p$, where $p \in S_{\mathrm{cl}}$ is the image of x. In particular, if Y is an algebraic scheme over $S = \operatorname{Spec}(k)$, where k is a field, not assumed algebraically closed, then $\mathbf{k}_x$ is finite algebraic over k. If $S = \operatorname{Spec}\mathbb{Z}$, then $\mathbf{k}_x$ is a finite field.

If Y is reduced, then we can identify $\mathcal{O}_X$ with a sheaf of functions with values $f(x) \in \mathbf{k}_x$, just as in the case of classical varieties but with the difference that the fields $\mathbf{k}_x$ vary from point to point. The reasoning is the same as before. If $f \in \mathcal{O}_Y(U)$ has $f(x) = 0$ for all $x \in U_{\mathrm{cl}}$, we want to show that $f = 0$. For this we can assume that $U = \operatorname{Spec}(R)$ is affine. The condition on f means that it belongs to the intersection of all maximal ideals in R, which is $\sqrt{0} = 0$ because R is Jacobson and reduced.

9.3. If $f\colon Y \to Y'$ is an S-morphism between schemes locally of finite type over S, then f is locally of finite type by Proposition 6.4. By Theorem 7.1 (ii), f maps Y_{cl} into Y'_{cl}. In other words, the map f between the underlying topological spaces of Y and Y' is a morphism in the category <u>JacSob</u> of Jacobson sober spaces defined in §3.6, and we have $f = \operatorname{Sob}(f_{\mathrm{cl}})$, where $f_{\mathrm{cl}}\colon Y_{\mathrm{cl}} \to Y'_{\mathrm{cl}}$ is the restriction of f to closed points.

As part of the morphism of ringed spaces f we also have a homomorphism of sheaves of rings $f^\flat\colon \mathcal{O}_{Y'} \to f_*\mathcal{O}_Y$. Setting $X = Y_{\mathrm{cl}}$, $X' = Y'_{\mathrm{cl}}$, there is a unique homomorphism $f^\flat_{\mathrm{cl}}\colon \mathcal{O}_{X'} \to (f_{\mathrm{cl}})_*\mathcal{O}_X$ corresponding to $f^\flat$ via the quasi-homeomorphisms $X \to Y$ and $X' \to Y'$. In general, this construction clearly extends the equivalence (T_1 spaces) $\cong$ <u>JacSob</u> in Proposition 3.6 to an equivalence (T_1 ringed spaces) $\cong$ (Jacobson sober ringed spaces and closed-point preserving morphisms).

9.4. Recall that a morphism $(f, f^\flat)\colon Y \to Y'$ of schemes is by definition a *local* morphism of locally ringed spaces, that is, each stalk homomorphism $f_p\colon \mathcal{O}_{Y',f(p)} \to \mathcal{O}_{Y,p}$ is a local homomorphism of local rings. Since the stalk $\mathcal{O}_{X,p}$ of $\mathcal{O}_X = i^{-1}\mathcal{O}_Y$ at each point $p \in X = Y_{\mathrm{cl}}$ coincides with $\mathcal{O}_{Y,p}$, and likewise for X', the morphism of T_1 ringed spaces $(f_{\mathrm{cl}}, f_{\mathrm{cl}}^\flat)\colon X \to X'$ corresponding to a morphism of schemes $f\colon Y \to Y'$ is local. Less obviously, the converse is also true.

Lemma. Let $(f, f^\flat)\colon Y \to Y'$ be a morphism of Jacobson sober ringed spaces such that $f(Y_{\mathrm{cl}}) \subseteq Y'_{\mathrm{cl}}$. Let $(g, g^\flat)\colon X \to X'$ be the corresponding morphism of T_1 ringed spaces, where $X = Y_{\mathrm{cl}}$, $X' = Y'_{\mathrm{cl}}$, $g = f_{\mathrm{cl}}$ and $f = \mathrm{Sob}(g)$.

Suppose Y and Y' are locally ringed spaces, hence so are X and X'. Then $(f, f^\flat)$ is a local morphism if and only if $(g, g^\flat)$ is.

Proof. At each closed point $p \in X = Y_{\mathrm{cl}}$, with $q = g(p) = f(p) \in X' = Y'_{\mathrm{cl}}$, the stalk homomorphism $f_p\colon \mathcal{O}_{Y',q} \to \mathcal{O}_{Y,p}$ coincides with $g_p\colon \mathcal{O}_{X',q} \to \mathcal{O}_{X,p}$. As we already observed, this shows that if $(f, f^\flat)$ is local, then $(g, g^\flat)$ is local.

For the converse, we can assume that $f_p\colon \mathcal{O}_{Y',f(q)} \to \mathcal{O}_{Y,p}$ is local at each closed point $p \in Y$, and we must prove that the same holds at every point. So now let $P \in Y$ be any point, with $Q = f(P) \in Y'$. Every element of in $\mathcal{O}_{Y',Q}$ is the germ s_Q of a section $s \in \mathcal{O}_{Y'}(U)$ on some open neighborhood U of Q. Let $V(s)$ be the locus of points $q \in U$ such that the germ s_q belongs to the maximal ideal of $\mathcal{O}_{Y',q}$. Recall that $V(s)$ is closed, since its complement is the set of points where s_q has a multiplicative inverse in $\mathcal{O}_{Y',q}$, and the inverse must be represented by a section such that $rs = 1$ on some neighborhood of q. To show that f_P is local, we assume that s_Q belongs to the maximal ideal of $\mathcal{O}_{Y',Q}$, that is, $Q \in V(s)$. Hence $V(s)$ contains $\overline{\{Q\}} \cap U$.

Let $W = f^{-1}(U)$. Then $f_P(s_Q)$ is the germ t_P of the section $t = f_U^\flat(s) \in \mathcal{O}_Y(W)$. The assumption that f_p is a local homomorphism for every closed point p implies that $V(t)$ contains every closed point in $f^{-1}(\overline{\{Q\}} \cap U)$. Now, W is Jacobson by Lemma 3.3, and $V(t)$ is closed in W, so this implies that $V(t)$ contains $f^{-1}(\overline{\{Q\}} \cap U)$. In particular, $P \in V(t)$, that is, $f_P(s_Q)$ belongs to the maximal ideal of $\mathcal{O}_{Y,P}$. $\square$

9.5. By Lemma 9.4, we now have an equivalence (T_1 locally ringed spaces) $\cong$ (Jacobson sober locally ringed spaces and closed-point preserving morphisms), where we require morphisms of locally ringed spaces to be local. This implies the following result.

Theorem. Let S be a Jacobson scheme. The functor sending a scheme Y to $X = Y_{\mathrm{cl}}$, $\mathcal{O}_X = i^{-1}\mathcal{O}_Y$ is an equivalence from the category of schemes locally of finite type over S and S-morphisms onto a full subcategory $\mathcal{C}$ of the category of T_1 locally ringed spaces over S_{cl} and local S_{cl}-morphisms.

The inverse functor sends an object X of $\mathcal{C}$ to the scheme $Y = (\mathrm{Sob}(X), i_*\mathcal{O}_X)$, which is a scheme over S since X is a locally ringed space over S_{cl}.

Remark. The functor $Y \mapsto Y_{\mathrm{cl}}$ also gives an equivalence from the category of all Jacobson schemes and morphisms locally of finite type to a subcategory of the category of T_1 locally ringed spaces, with left inverse $X \mapsto \mathrm{Sob}(X)$. In this generality, however, the image category is not a *full* subcategory of T_1 locally ringed spaces, as the morphism of Jacobson schemes

$f\colon Y \to Y'$ corresponding to an arbitrary morphism $g\colon X \to X'$ between the T_1 locally ringed spaces $X = Y_{\mathrm{cl}}$ and $X' = Y'_{\mathrm{cl}}$ need not be locally of finite type.

Restricting to the category of schemes locally of finite type over a Jacobson base scheme S resolves this difficulty, since if g is a morphism of locally ringed spaces over S_{cl}, then f is an S-morphism, hence automatically locally of finite type.