VARIETIES AS SCHEMES

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0.1. Classical algebraic geometry is the study of algebraic varieties, meaning spaces that can be described locally as solution sets of polynomial equations over an algebraically closed field, such as the complex numbers $\mathbb{C}$.

Grothendieck's *Éléments de Géometrie Algébrique* (EGA) reformulated the foundations of the subject as the study of *schemes*. The language of schemes has since come to be universally accepted because of its advantages in both technical simplicity and conceptual clarity over older theories of varieties. At its heart, however, algebraic geometry is still primarily the study of algebraic varieties. The most important examples of schemes either arise directly from varieties, or are closely related to them. In order to apply scheme theory to varieties, or indeed to get any real idea of what schemes are are all about, one must first understand *how classical varieties are schemes.*

0.2. The purpose of these notes is to state precisely and prove the equivalence

$$\text{classical algebraic varieties over } k = \text{reduced algebraic schemes over } k,$$

where k is an algebraically closed field. The term *algebraic scheme* means a scheme locally of finite type over a field.[1]

0.3. A full statement of the equivalence in §0.2 will be given in Theorem 9.1. To help the reader make sense of the definitions and results leading up to Theorem 9.1, we preview here the essential features of the equivalence, omitting many details.

Let k be an algebraically closed field. A classical *affine* algebraic variety X over k is determined by the ring of polynomial functions $R = R(X)$ on X. The corresponding reduced algebraic k-scheme in this case will be the affine scheme $Y = \operatorname{Spec}(R)$. We will see in Proposition 7.10 that this construction provides natural equivalences of categories between (a) classical affine varieties X over k, (b) finitely-generated reduced k-algebras R (with arrows reversed in the category of k-algebras), and (c) reduced algebraic affine k-schemes $Y = \operatorname{Spec}(R)$. The equivalence between (a) and (b) is a consequence of *Hilbert's Nullstellensatz*, which we will prove in Lemma 7.7. The equivalence between (b) and (c) is a special case of a basic property of affine schemes.

It is also possible to pass between a classical affine variety X and the corresponding scheme $Y = \operatorname{Spec}(R)$ directly, without going through the ring $R = R(X)$ as an intermediate step, in the following way.

Date: May 14, 2019.

[1]In EGA, an algebraic scheme (or 'prescheme') is also required to be quasi-compact. We do not require this here. It will follow from the construction that the scheme in the equivalence is quasi-compact if and only if the corresponding variety is.

Given Y, the variety X can be identified with the subspace Y_{cl} of closed points of Y, which (as we will prove) is also the image in Y of the set of k-points $Y(k)$. Sections of the structure sheaf $\mathcal{O}_Y$ of the scheme Y can be evaluated at k-points to give k-valued functions on open subsets of X. This turns out to give the sheaf of regular functions $\mathcal{O}_X$ on X.

Given X, the underlying space of the scheme $Y = \mathrm{Spec}(R(X))$ is the *sober space* $\mathrm{Sob}(X)$, whose points correspond to irreducible closed subsets of X. There is a canonical inclusion map $i\colon X \hookrightarrow Y = \mathrm{Sob}(X)$ and we will see that $\mathcal{O}_Y$ is isomorphic to $i_*\mathcal{O}_X$.

When X and Y are not assumed to be affine, the equivalence is given by a pair of functors $\mathrm{Cl}(-)$ and $\mathrm{Sob}(-)$ which generalize the correspondence we just described in the affine case. Given Y, the variety $X = \mathrm{Cl}(Y)$ is the space of closed points of Y (which are also the k-points of Y), with sheaf of regular functions $\mathcal{O}_X$ defined by evaluating sections of $\mathcal{O}_Y$ at k-points. Given X, the scheme Y is its soberization $(Y, \mathcal{O}_Y) = (\mathrm{Sob}(X), i_*\mathcal{O}_X)$.

Using affine coverings, the proof that the functors $\mathrm{Cl}(-)$ and $\mathrm{Sob}(-)$ are well-defined and inverse to one other reduces to the affine case.

0.4. We presume that the reader is familiar with the elementary theory of schemes, including the definitions of a scheme and a morphism of schemes, the construction of open and closed subschemes of a scheme, and the notions of scheme and morphism over a base scheme S (abbreviating these as usual to "k-scheme" and "k-morphism" when $S = \mathrm{Spec}(k)$).

We take as known the theorem that the ring of global functions $\mathcal{O}_Y(Y)$ on an affine scheme $Y = \mathrm{Spec}(R)$ is equal to R, and the characterization of morphisms $X \to \mathrm{Spec}(R)$ as corresponding to ring homomorphisms $R \to \mathcal{O}_X(X)$. We also take as known the theorem that every fiber $f^{-1}(p)$ of a morphism $f\colon X \to Y$, considered as a subspace of X, is homeomorphic to the underlying space of the scheme-theoretic fiber $\mathrm{Spec}(\mathbf{k}_p) \times_Y X$.

For the proof of the equivalence in §0.2, we will need some additional results, such as the theory of Jacobson schemes and morphisms locally of finite type. We will develop these here. Our approach generally follows that of EGA IV, Ch. 10 and (for sober spaces) the 1971 revised edition of EGA I.

In order to have anything to prove, we also need some facts about classical varieties, independent of their interpretation as schemes. To this end we give in §1 a brief introduction to varieties, using the definition due to Serre in *Faisceaux Algébriques Cohérents*.

0.5. Conventions: all rings are commutative, with unit. All ring homomorphisms are unital.

1. Classical varieties

In this section we develop just enough of the theory of classical algebraic varieties to give meaning to the left hand side of the equivalence in §0.2.

In the process, we will also examine how a classical affine variety X with coordinate ring $R(X)$ is related to the reduced algebraic affine k-scheme $Y = \mathrm{Spec}(R(X))$, which will turn out to correspond to X via the equivalence. We will see, in fact, that Y completely determines X, in a natural manner. In §7, we will combine this construction of X from Y with the property that Y is a Jacobson scheme to establish the affine case of the equivalence in §0.2.

1.1. Let k be an algebraically closed field. A classical *affine algebraic variety* X is the solution locus in k^n of a system of polynomial equations. Since an equation $f(x) = g(x)$ can be written $f(x) - g(x) = 0$, we may also define X to be the zero locus

$$V(F) = \{a \in k^n \mid f(a) = 0 \text{ for all } f \in F\}$$

of a set of polynomials $F \subseteq k[\mathbf{x}] = k[x_1, \ldots, x_n]$.

1.2. The identities

$$\emptyset = V(\{1\}), \quad k^n = V(\{0\}), \quad V(F) \cup V(G) = V(F \cdot G), \quad \bigcap_\alpha V(F_\alpha) = V(\bigcup_\alpha F_\alpha)$$

show that the affine varieties $X \subseteq k^n$ are the closed subsets of a topology on k^n, called the *Zariski topology*. For this reason, we also refer to an affine variety $X \subseteq k^n$ as a *closed subvariety* of k^n. The Zariski topology on X is defined to be its subspace topology in k^n. The closed subsets of X in the Zariski topology are thus the closed subvarieties $Y \subseteq k^n$ that are contained in X.

1.3. For any subset $Z \subseteq k^n$, the set $\mathcal{I}(Z) = \{f \mid Z \subseteq V(f)\}$ of polynomials vanishing on Z is an ideal in the polynomial ring $k[\mathbf{x}] = k[x_1, \ldots, x_n]$. Since $F = \mathcal{I}(Z)$ is the largest subset of $k[\mathbf{x}]$ such that $Z \subseteq V(F)$, we see that $X = V(\mathcal{I}(Z))$ is the smallest closed subvariety containing Z, that is, X is the closure $\overline{Z}$ of Z in the Zariski topology. In particular, for every closed subvariety $X \subseteq k^n$, we have $X = V(\mathcal{I}(X))$. This shows that the correspondence

$$X \mapsto \mathcal{I}(X), \quad I \mapsto V(I)$$

is an order-reversing bijection between closed subvarieties $X \subseteq k^n$ and ideals $I \subseteq k[\mathbf{x}]$ of the form $I = \mathcal{I}(X)$. According to Hilbert's Nullstellensatz, the latter are exactly the radical ideals $I = \sqrt{I}$ in $k[\mathbf{x}]$. We will not make any use of the Nullstellensatz at this stage. We will prove it later (Lemma 7.7) as part of the equivalence between varieties and schemes.

1.4. Given an affine variety $X \subseteq k^n$, there is a k-algebra homomorphism from $k[\mathbf{x}]$ to the ring of k-valued functions on X, given by evaluating polynomials $f \in k[\mathbf{x}]$ at the points of X. The image of this homomorphism is denoted $R(X)$ and called the *ring of polynomial functions* on X, or the *coordinate ring* of X.

By definition, $\mathcal{I}(X)$ is the kernel of the homomorphism $k[\mathbf{x}] \to R(X)$. Hence we have a canonical isomorphism $R(X) \cong k[\mathbf{x}]/\mathcal{I}(X)$.

Since $R(X)$ is a ring of functions on X, we can define the vanishing locus $V(F) \subseteq X$ for any subset $F \subseteq R(X)$, and the ideal $\mathcal{I}(Z) \subseteq R(X)$ for any $Z \subseteq X$. Then $V(F) \subseteq X$ is always closed, and every closed subvariety $Z \subseteq X$ has the form $Z = V(I)$, where $I = \mathcal{I}(Z)$. These are corollaries to the corresponding facts in the case $X = k^n$.

1.5. *Remark.* It is true, but not obvious, that the ideal of $X = k^n$ as a closed subvariety of k^n is $\mathcal{I}(X) = 0$, so $R(X) = k[\mathbf{x}]$. Equivalently, if a polynomial $f \in k[\mathbf{x}]$ evaluates to $f(a) = 0$ for all $a \in k^n$, then f is the zero polynomial.

This is a consequence of Hilbert's Nullstellensatz, or of the correspondence between varieties and schemes, but it can also be proven in a more elementary way by reducing to the case $n = 1$ and using the fact that k^1 is infinite (because k is algebraically closed), while every non-zero polynomial $f(x)$ in one variable has finitely many roots.

1.6. The classical affine space k^n can be naturally identified with the set of points $p \in \mathbb{A}^n_k = \operatorname{Spec}(k[x_1,\dots,x_n])$ whose residue field is $\mathbf{k}_p = k$. In fact, these are precisely the points p corresponding to maximal ideals in $k[\mathbf{x}]$ (that is, closed points of $\mathbb{A}^n_k$) of the form $\mathfrak{m} = \mathcal{I}(a) = \ker(\mathrm{ev}_a)$, where $\mathrm{ev}_a \colon k[\mathbf{x}] \to k$ is the evaluation homomorphism $\mathrm{ev}_a(f) = f(a)$ at a point $a \in k^n$.

More generally, if Y is a scheme over a field k, a k-morphism $\operatorname{Spec}(k) \to Y$ is called a *k-point* of Y. Let $Y(k)$ denote the set of k-points of Y. The map sending each k-point to its image in Y is a bijection from $Y(k)$ onto the set of points $p \in Y$ such that $\mathbf{k}_p = k$. In the case $Y = \mathbb{A}^n_k$, the k-points correspond to k-algebra homomorphisms $\varphi \colon k[\mathbf{x}] \to k$, and every such homomorphism is given by $\varphi = \mathrm{ev}_a$ for some $a \in k^n$, namely $a = (a_1,\dots,a_n)$ where $a_i = \varphi(x_i)$. This gives the identification $k^n \cong \mathbb{A}^n_k(k) \hookrightarrow \mathbb{A}^n_k$ described above.

Given any ideal $I \subseteq k[\mathbf{x}]$, we have on the one hand an affine variety $X = V(I) \subseteq k^n$, as in §1.3, and on the other a closed subset $V(I) \subseteq \mathbb{A}^n_k$. Here we are using the same notation $V(I)$ to mean two different things. However, since $a \in X$ if and only if $I \subseteq \ker(\mathrm{ev}_a)$, it is clear that under our identification of k^n with the image of $\mathbb{A}^n_k(k)$ in $\mathbb{A}^n_k$, we have $X = V(I) \cap k^n$ in $\mathbb{A}^n_k$. In other words, the classical Zariski topology on k^n is the same as its subspace topology in the scheme $\mathbb{A}^n_k$.

1.7. If $X \subseteq k^n$ is an affine variety, then $Y = \operatorname{Spec}(R(X)) = \operatorname{Spec}(k[\mathbf{x}]/\mathcal{I}(X))$ is a closed subscheme of $\mathbb{A}^n_k$. Under the identifications in §1.6, the set of k-points $Y(k)$ is the subset of $\mathbb{A}^n_k(k)$ corresponding to $X \subseteq k^n$. Thus we have a canonical bijection $X \cong Y(k)$ such that the Zariski topology on X coincides with the subspace topology on the image of $Y(k)$ in the affine scheme $Y = \operatorname{Spec}(R(X))$.

Note that $R(X)$ is always a reduced k-algebra, since it is a ring of functions with values in the field k, and a nilpotent function is obviously zero. Hence $\mathcal{I}(X)$ is always a radical ideal, and $Y = \operatorname{Spec}(R(X))$ is a reduced k-scheme. Since $R(X)$ is a finitely generated k-algebra, Y is a scheme of finite type over k (see §2). Thus Y is a reduced algebraic affine k-scheme.

1.8. Let $X \subseteq k^m$ and $Y \subseteq k^n$ be affine varieties, with coordinates $x_1,\dots,x_m$ on X and $y_1,\dots,y_n$ on Y. A *polynomial map* $\varphi \colon X \to Y$ is a map such that the coordinates y_i of $\varphi(a)$ are given by polynomial functions $f_i(a)$ in the coordinates of $a \in X$. We can represent the functions f_i by polynomials $f_i \in k[\mathbf{x}]$, although the map φ is of course determined by the images of the f_i in $R(X)$.

Any tuple $(f_1,\dots,f_n)$ of polynomial functions $f_i \in R(X)$ defines a polynomial map $\varphi \colon X \to k^n$. The map φ induces a k-algebra homomorphism $\psi \colon k[\mathbf{y}] \to R(X)$ sending $g \in k[\mathbf{y}]$ to the function $g \circ \varphi$ on X. More explicitly, the homomorphism ψ is given on the generators of $k[\mathbf{y}]$ by $\psi(y_i) = f_i$.

The condition to have $\varphi(X) \subseteq Y$ is just that $\psi(g) = 0$ for all $g \in \mathcal{I}(Y)$; in other words, that ψ factors through a k-algebra homomorphism $R(Y) \to R(X)$. Hence polynomial maps $\varphi \colon X \to Y$ correspond bijectively to k-algebra homomorphisms $\psi \colon R(Y) \to R(X)$, the relationship between φ and ψ being given by $\psi(g) = g \circ \varphi$ for all $g \in R(Y)$.

Equivalently, polynomial maps $\varphi \colon X \to Y$ correspond bijectively to k-morphisms of k-schemes $\varphi' \colon X' = \operatorname{Spec}(R(X)) \to \operatorname{Spec}(R(Y)) = Y'$. It is easy to see that under the identifications $X = X'(k)$, $Y = Y'(k)$, the map φ is just the map on k-points induced by φ'. This implies in particular that every polynomial map φ is continuous in the Zariski topology (which is also not hard to verify directly).

1.9. Provisionally, we might like to define a morphism $\varphi\colon X \to Y$ of affine varieties to be a polynomial map. Although this characterization of morphisms is correct in the affine case, it is unsuitable as a definition, since it doesn't extend naturally to other classical varieties—such as projective or quasi-projective varieties, or open subsets of affine varieties—which may not be affine in general.

The right way to define a morphism of classical varieties is analogous to what we do for schemes, or for ringed spaces arising in other flavors of geometry, such as smooth manifolds. Namely, we should first define the *sheaf of regular functions* on an affine variety, and then use this to define general (not necessarily affine) varieties and morphisms of varieties.

We now give the relevant definitions.

1.10. If X is a classical affine variety and $f \in R(X)$ is a polynomial function, we define $X_f = X - V(f)$. Then X_f is an open subset of X, and since every closed subset $Z = V(I) \subseteq X$ is the intersection of closed subsets $V(f)$ for $f \in I$, it follows that every open subset $U \subseteq X$ is a union of open subsets of the form X_f. Thus the X_f form a base of open sets for the Zariski topology on X.

By construction, the function f has no zeroes on X_f, so the function $1/f$ exists on X_f.

Definition. Let $X \subseteq k^n$ be an affine variety. A function $f\colon U \to k$ on an open subset $U \subseteq X$ is *regular* if U can be covered by open subsets of the form $U_h = U \cap X_h$ on which $f = g/h$, for some $g, h \in R(X)$.

The *sheaf of regular functions* $\mathcal{O}_X$ on X is defined by taking $\mathcal{O}_X(U)$ to be the set of regular functions on U.

For clarity, note that the restriction of a regular function $f \in \mathcal{O}_X(U)$ to any open subset $V \subseteq U$ is regular on V; this makes $\mathcal{O}_X$ a presheaf, and the local nature of the definition of regular function implies that it is a sheaf.

Using the fact that $X_f \cap X_g = X_{fg}$, one can verify that that $\mathcal{O}_X(U)$ is a subring of the ring $\mathcal{F}_X(U)$ of all k-valued functions on U. Thus $\mathcal{O}_X$ is a subring sheaf of the sheaf $\mathcal{F}_X$ of all k-valued functions on (open subsets of) X.

1.11. Let us compare the sheaf of regular functions $\mathcal{O}_X$ on a classical affine variety $X \subseteq k^n$ with the structure sheaf $\mathcal{O}_Y$ of the affine scheme $Y = \operatorname{Spec}(R(X))$.

As in §1.7, we can identify X as a topological space with the image of $Y(k)$ in Y, or equivalently with the subspace of Y consisting of points $p \in Y$ such that $\mathbf{k}_p = k$. Given an open subset $U \subseteq Y$ and $f \in \mathcal{O}_Y(U)$, we can evaluate f at each k-point $p \in U$ to get a k-valued function on $X \cap U$. Denoting the inclusion map by $i\colon X \cong Y(k) \hookrightarrow Y$, this gives a homomorphism of sheaves of rings $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{F}_X$. Using the fact that i^{-1} is left adjoint to i_*, we have a corresponding homomorphism $i^\sharp\colon i^{-1}\mathcal{O}_Y \to \mathcal{F}_X$, whose image is the subsheaf of $\mathcal{F}_X$ consisting of functions given locally by evaluating sections of $\mathcal{O}_Y$.

On $U = Y$, the map $i^\flat_Y\colon R(X) = \mathcal{O}_Y(Y) \to \mathcal{F}_X(X)$ is just evaluation of elements $f \in R(X)$ as functions on X. More generally, on $U = Y_f$, we have $X \cap U = X_f$, so f is invertible in $\mathcal{F}_X(X_f)$, and $i^\flat_U\colon R(X)_f = \mathcal{O}_Y(Y_f) \to \mathcal{F}_X(X_f)$ is the unique extension of the evaluation map $R(X) \to \mathcal{F}_X(X_f)$ to the localized ring $R(X)_f$.

The sections of $\mathcal{O}_Y$ on any open set are given locally by elements of $R(X)_h$ on subsets of the form Y_h. Hence we can rephrase the definition of the sheaf of regular functions on X to say that $\mathcal{O}_X$ is the image of the homomorphism $i^\sharp\colon i^{-1}\mathcal{O}_Y \to \mathcal{F}_X$.

Although the preceding discussion may give the impression that the sheaf of regular functions on a classical affine variety X has been defined so as to match the definition of the structure sheaf $\mathcal{O}_Y$ on the affine scheme $Y = \operatorname{Spec}(R(X))$, we should remark that in truth, the motivation is the other way around. Serre had originally defined the sheaf of regular functions on a classical affine variety, and Grothendieck subsequently generalized this to define the structure sheaf $\mathcal{O}_Y$ of an affine scheme $Y = \operatorname{Spec}(R)$.

1.12. *Definition.* A *classical algebraic variety over k* is a topological space X, equipped with a sheaf $\mathcal{O}_X$ of rings of k-valued functions (that is, a subring sheaf $\mathcal{O}_X \subseteq \mathcal{F}_X$), such that X can be covered by open sets U for which $(U, \mathcal{O}_X|U)$ is isomorphic to a classical affine variety with its sheaf of regular functions.

A *morphism of classical algebraic varieties* $f\colon X \to Y$ is a continuous map such that the canonical homomorphism $\mathcal{F}_Y \to f_*\mathcal{F}_X$ sending $g \in \mathcal{F}_Y(U)$ to $g \circ f \in \mathcal{F}_X(f^{-1}(U))$ carries $\mathcal{O}_Y$ into $f_*\mathcal{O}_X$. In other words, if g is a regular function on $U \subseteq Y$, then we require $g \circ f$ to be regular on $f^{-1}(U)$.

These definitions easily imply that every polynomial map $f\colon X \to X'$ between classical affine varieties is a morphism. However, we cannot yet conclude that, conversely, every morphism between classical affine varieties is given by a polynomial map, although we will eventually prove this (Corollary 7.6).

To prove the converse, we will first need to know that every global regular function on X is a polynomial function, *i.e.*, that $R(X) = \mathcal{O}_X(X)$, or that the sheaf homomorphism $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{O}_X$ corresponding to the homomorphism $i^\sharp\colon i^{-1}\mathcal{O}_Y \to \mathcal{O}_X \subseteq \mathcal{F}_X$ from §1.11 gives a surjective homomorphism $i^\flat_Y\colon R(X) \to \mathcal{O}_X(X)$ on global sections. For this we will need to prove that $i^\flat$ is actually an isomorphism $\mathcal{O}_Y \cong i_*\mathcal{O}_X$ (Lemma 7.3).

2. Morphisms locally of finite type

2.1. *Definition.* A morphism of schemes $f\colon X \to Y$ is *locally of finite type* if for every point $x \in X$, there exist open affine neighborhoods $U = \operatorname{Spec}(B)$ of x in X and $V = \operatorname{Spec}(A)$ of $f(x)$ in Y, such that $f(U) \subseteq V$ and the ring homomorphism $A \to B$ corresponding to the morphism $(f|U)\colon U \to V$ makes B a finitely generated A-algebra.

2.2. Suppose $U = \operatorname{Spec}(B)$ and $V = \operatorname{Spec}(A)$ satisfy the conditions in Definition 2.1. We can replace U with any smaller basic open neighborhood $U_g = \operatorname{Spec} B[g^{-1}]$ of x, since $B[g^{-1}]$ is a finitely generated B-algebra, hence a finitely generated A-algebra.

We can also replace V with any smaller open affine $V' = \operatorname{Spec}(A')$ such that $f(U) \subseteq V' \subseteq V$. If we do this, the morphisms $U \to V' \hookrightarrow V$ correspond to ring homomorphism $A \to A' \to B$, and any finite set of generators for B as an A-algebra also generates B as an A'-algebra. Note that we can take V' to be an arbitrary affine neighborhood of $f(x)$ in V if we also replace U with a basic open U_g such that $U_g \subseteq f^{-1}(V')$.

It follows that the definition is local on both X and Y. More precisely: (i) if $f\colon X \to Y$ is locally of finite type, then so is $(f|U)\colon U \to V$, for all open subschemes $U \subseteq X$ and $V \subseteq Y$ such that $f(U) \subseteq V$; (ii) if Y has a covering by open subschemes V_α such that $f^{-1}(V_\alpha) \to V_\alpha$ is locally of finite type for all α, then f is locally of finite type; and (iii) if X has a covering by open subschemes U_α such that $(f|U_\alpha)\colon U_\alpha \to Y$ is locally of finite type for all α, then f is locally of finite type.

In particular, since the identity morphism on any scheme is obviously locally of finite type, it follows that every open embedding is locally of finite type. It is also easy to see that every closed embedding is locally of finite type.

2.3. *Proposition.* If $f\colon X \to Y$ and $g\colon Y \to Z$ are locally of finite type, then so is $g \circ f$.

Proof. Given $x \in X$, let $y = f(x)$ and $z = g(y)$. We can find affine open neighborhoods $y \in V = \operatorname{Spec}(B) \subseteq Y$ and $z \in W = \operatorname{Spec}(A) \subseteq Z$ such that $g(V) \subseteq W$ and B is a finitely generated A-algebra. Using the observations in §2.2, we can find a basic open neighborhood $V_h = \operatorname{Spec}(B[h^{-1}]) \subseteq V$ of y and an affine neighborhood $U = \operatorname{Spec}(C) \subseteq X$ of x such that $f(U) \subseteq V_h$ and C is a finitely generated $B[h^{-1}]$-algebra. We now have $A \to B \to B[h^{-1}] \to C$ with each ring a finitely generated algebra over the previous one, so C is a finitely generated A-algebra. □

2.4. *Proposition.* Given $f\colon X \to Y$ and $g\colon Y \to Z$, if $g \circ f$ is locally of finite type, then so is f.

Proof. Given $x \in X$, let $y = f(x)$ and $z = g(y)$. We can find $x \in U = \operatorname{Spec}(C) \subseteq X$ and $z \in W = \operatorname{Spec}(A) \subseteq Z$ such that $(g \circ f)(U) \subseteq W$ and C is a finitely generated A-algebra. Choose any open affine neighborhood $V = \operatorname{Spec}(B) \subseteq Y$ of y such that $g(V) \subseteq W$. Then we can find a basic open subset $U_h \subseteq U$ such that $x \in U_h \subseteq f^{-1}(V)$. The morphisms $U_h \to V \to W$ correspond to ring homomorphisms $A \to B \to C[h^{-1}]$. Since C is a finitely generated A-algebra, so is $C[h^{-1}]$. Hence $C[h^{-1}]$ is also a finitely generated B-algebra. □

This proposition implies, in particular, that every S-morphism between schemes locally of finite type over S is locally of finite type.

2.5. *Proposition.* If $f\colon X \to Y$ is locally of finite type, then for *all* open affines $U = \operatorname{Spec}(B) \subseteq X$ and $V = \operatorname{Spec}(A) \subseteq Y$ such that $f(U) \subseteq V$, B is a finitely generated A-algebra. In particular, if $\varphi\colon A \to B$ is a ring homomorphism for which the corresponding morphism of affine schemes $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is locally of finite type, then B is finitely generated as an A-algebra.

Proof. By §2.2, the morphism $(f|U)\colon U \to V$ is locally of finite type; moreover, we can find pairs $U_g = \operatorname{Spec}(B[g^{-1}]) \subseteq U$ and $V_h = \operatorname{Spec}(A[h^{-1}]) \subseteq V$ such that $f(U_g) \subseteq V_h$ and $B[g^{-1}]$ is a finitely generated $A[h^{-1}]$-algebra, hence also a finitely generated A-algebra, with the open sets U_g in these pairs covering U. Since U is quasi-compact, we can take the covering $U = \bigcup_i U_{g_i}$ to be finite.

Thus we now have an A-algebra B and elements $g_1, \ldots, g_n \in B$ such that $B[g_i^{-1}]$ is a finitely generated A-algebra for each i, and the g_i generate the unit ideal in B. We are to show that B is a finitely generated A-algebra. Let

$$a_1 g_1 + \cdots + a_n g_n = 1.$$

For each $i = 1, \ldots, n$, we can take $B[g_i^{-1}]$ to be generated as an A-algebra by g_i^{-1} and the images in $B[g_i^{-1}]$ of finitely many elements $x_{ij} \in B$. Let B' be the A-subalgebra of B generated by all the elements a_i, g_i, and x_{ij}. Let $U' = \operatorname{Spec}(B')$ and consider the sheaf $\widetilde{B}$ on U' associated to the B'-module B.

The inclusion $B' \hookrightarrow B$ induces a sheaf homomorphism $\mathcal{O}_{U'} \hookrightarrow \widetilde{B}$. We have $B'[g_i^{-1}] = B[g_i^{-1}]$, since B' contains all the x_{ij}. Hence the homomorphism $\mathcal{O}_{U'} \hookrightarrow \widetilde{B}$ restricts to an isomorphism on U'_{g_i}. The g_i generate the unit ideal in B', with the same coefficients a_i as in B, so the open subsets U'_{g_i} cover U'. It follows that $\mathcal{O}_{U'} \hookrightarrow \widetilde{B}$ is an isomorphism, hence so is $B' \hookrightarrow B$, that is, $B' = B$. But B' is a finitely generated A-algebra by construction. $\square$

3. Finite morphisms

3.1. *Definition.* A morphism of affine schemes $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is *finite* if the corresponding ring homomorphism $R \to S$ makes S an R-algebra finitely generated as an R-module.

We omit the definition of a finite morphism of general schemes, as we only need the affine case for now.

3.2. Let S be an R-algebra. An element $s \in S$ is *integral* over R if there is a monic polynomial $f(x) \in R[x]$ such that $f(s) = 0$. If S is generated as an R-algebra by an integral element s, then we can use the relation $f(s) = 0$ to reduce any power s^n, where $n \geq \deg(f)$, to an R-linear combination of smaller powers of s. It follows that S is generated as an R-module by the finite set of powers s^n for $n < \deg(f)$.

This generalizes easily to show that S is finitely generated as an R-module if it is generated as an R-algebra by a finite set of elements integral over R, but here we will only need the case of a single integral element.

3.3. *Proposition* (Nakayama's lemma). Let R be a local ring with maximal ideal P, and let M be a finitely generated R-module. If $M/PM = 0$, then $M = 0$.

For completeness, we give the standard proof:

Proof. Suppose $x_1, \ldots, x_n$ generate M. By hypothesis, $PM = M$, so each x_i belongs to PM. Hence we can write $x_i = \sum_j a_{ij} x_j$, with all $a_{ij} \in P$. Let A be the $n \times n$ matrix with entries a_{ij}. By construction, we have $(I - A)v = 0$, where $v \in M^n$ is the vector with entries x_j. The determinant $\det(I - A) \in R$ is congruent to 1 modulo P, hence it is a unit in R, so the matrix $I - A$ is invertible. Hence all the x_j are zero, that is, $M = 0$. $\square$

The next results can be viewed as geometric consequences of Nakayama's Lemma.

3.4. *Proposition.* Let R be a subring of S such that S is a finitely-generated R module. Then the finite morphism $f\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is surjective.

Proof. Given $P \in \operatorname{Spec}(R)$, the fiber $f^{-1}(P)$ can be identified with the underlying space of the scheme $\operatorname{Spec}(\mathbf{k}_P \otimes_R S) = \operatorname{Spec}(S_P/PS_P)$. Since $R \to S$ is injective, so is the localization $R_P \to S_P$. In particular, $S_P \neq 0$. Since f is finite, S_P is a finitely generated R_P-module (generated by the image of any generating set for S as an R-module). By Nakayama's lemma, it follows that $S_P/PS_P \neq 0$, so $f^{-1}(P)$ is non-empty. $\square$

3.5. *Corollary.* If $f\colon X = \operatorname{Spec}(S) \to \operatorname{Spec}(R) = Y$ is a finite morphism of affine schemes, then $f(Z)$ is closed for every closed subset $Z \subseteq X$. In particular, if $z \in X$ is a closed point, then $f(z)$ is a closed point of Y.

Proof. Let $\varphi\colon R \to S$ be the ring homomorphism corresponding to f. Let $Z = V(I)$, where $I \subseteq S$ is an ideal. Then the closure of $f(Z)$ is $V(J)$, where $J = \varphi^{-1}(I)$. Now R/J is a subring of S/I, and S/I is a finitely-generated (R/J)-module, so $Z \to \overline{f(Z)}$ is surjective by Lemma 3.4.

4. Geometric properties of $\mathbb{A}^1_R$

For any ring R, the 'affine line' $\mathbb{A}^1_R = \mathrm{Spec}(R[x])$ is a scheme over $\mathrm{Spec}(R)$, with structure morphism $\pi\colon \mathbb{A}^1_R \to \mathrm{Spec}(R)$ corresponding to the inclusion $R \hookrightarrow R[x]$. Here we develop some geometric properties of this morphism, for use in the proof of the basic theorem on Jacobson schemes, Theorem 6.1.

4.1. We begin by describing the topology of $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$ in the case that k is a field.

As usual, a *closed point* in any topological space is a point p such that $\{p\}$ is closed. If $\{p\}$ is locally closed, we say that p is a *locally closed point*.

The prime ideals of $k[x]$ are the zero ideal $\mathfrak{p} = (0)$ and the maximal ideals $\mathfrak{m} = (g(x))$, where $g(x) \in k[x]$ is an irreducible polynomial. The points $\mathfrak{m}$ are closed, while $\mathfrak{p}$ is the generic point, whose closure $\overline{\{\mathfrak{p}\}} = V(0)$ is the whole space $\mathbb{A}^1_k$. Every non-zero ideal of $k[x]$ is a principal ideal $(f(x))$, with $V(f)$ consisting of the closed points corresponding to irreducible factors of f. Hence every proper closed subset of $\mathbb{A}^1_k$ is a finite set of closed points.

The whole space $\mathbb{A}^1_k$ is infinite. This is obvious if k is infinite, but it is also true if k is a finite field, since there are irreducible polynomials over k of every degree. The generic point $\mathfrak{p}$ is not locally closed, since this would mean that $\{\mathfrak{p}\}$ is open in its closure, and therefore open. But $\{\mathfrak{p}\}$ is not open, since its complement is not a finite set of closed points.

4.2. *Proposition.* For any ring R, let P be a locally closed point of $\mathbb{A}^1_R$, and let $Q = \pi(P)$, where $\pi\colon \mathbb{A}^1_R \to \mathrm{Spec}(R)$ is the structure morphism. Then:

(i) P is a closed point of the fiber $\pi^{-1}(Q)$, and

(ii) the residue field $\mathbf{k}_P$ is a finite algebraic extension of $\mathbf{k}_Q$, of the form $\mathbf{k}_Q[x]/(g(x))$, where $g \in \mathbf{k}_Q[x]$ is an irreducible polynomial.

Proof. Since P is locally closed in $\mathbb{A}^1_R$, it is also locally closed in the the fiber $\pi^{-1}(Q)$, which is homeomorphic to the underlying space of the scheme-theoretic fiber $\mathrm{Spec}(\mathbf{k}_Q \otimes_R R[x]) = \mathrm{Spec}(\mathbf{k}_Q[x]) = \mathbb{A}^1_{\mathbf{k}_Q}$. We saw in §4.1 that the generic point of $\mathbb{A}^1_{\mathbf{k}_Q}$ is not locally closed, so P is a closed point of $\mathbb{A}^1_{\mathbf{k}_Q}$. The residue field $\mathbf{k}_P$ of P in $\mathbb{A}^1_R$ is the same as its residue field in $\mathbb{A}^1_{\mathbf{k}_Q}$, and the latter has the desired form $\mathbf{k}_Q[x]/(g(x))$. □

4.3. *Proposition.* For any ring R, let $\pi\colon \mathbb{A}^1_R \to \mathrm{Spec}(R)$ be the structure morphism. If P is a locally closed point of $\mathbb{A}^1_R$, then $Q = \pi(P)$ is a locally closed point of $\mathrm{Spec}(R)$.

Proof. Let $S = R[x]/P$ and $T = R/Q$. We can identify $\mathrm{Spec}(S)$ with the closure $\overline{\{P\}} = V(P)$ of $\{P\}$ in $\mathbb{A}^1_R$, with the point P corresponding to the zero ideal $(0) \subseteq S$. Similarly, $\mathrm{Spec}(T)$ is the closure of $\{Q\}$ in $\mathrm{Spec}(R)$. The ring S is an integral domain, with $T \subseteq S$ a subring. Let K be the fraction field of S and $L \subseteq K$ the fraction field of T. Then $K = \mathbf{k}_P$ and $L = \mathbf{k}_Q$.

Since P is locally closed, $\{P\}$ is an open subset in $\mathrm{Spec}(S)$, which we can take to be $\mathrm{Spec}(S)_f$ for some $f \in S$. Then S_f is a field, that is, $S_f = K$, since (0) is its unique prime ideal. To prove that Q is locally closed, we need to find an element $a \in T$ such that T_a is a field.

By Proposition 4.2(ii), we have $K = L[x]/(g(x))$ for some irreducible polynomial g over L. On the other hand, the L-subalgebra of K generated by x is the subring $LS \subseteq K$, which is the localization $(T^{\times})^{-1}S$ of S. Thus $LS = K$, so we can write $f^{-1} \in K$ as $f^{-1} = s/b$, where $s \in S$ and $b \in T$. Then the T_b-subalgebra of K generated by x contains both S and f^{-1}, so x generates $K = S_f$ as a T_b-algebra.

Multiplying the polynomial g such that $K = L[x]/(g(x))$ by some element of T if needed, we can clear denominators in the coefficients of g, and assume that $g \in T[x]$. In this form, g need not be monic, so let $c \in T$ be its leading coefficient, that is, $c \neq 0$ and $g(x) = cx^n + (\text{lower degree terms})$.

Let $T_a = T_{bc}$, so T_a contains both T_b and T_c. The monic polynomial $h(x) = c^{-1}g(x)$ has coefficients in $T_c \subseteq T_a$, and $h(x) = 0$ in K, so the element $x \in K$ is integral over T_a. Furthermore, x generates K as an algebra over T_b and hence also over T_a. It follows by §3.2 that K is a finitely generated T_a-module. Corollary 3.5 then implies that the zero ideal is a closed point of $\operatorname{Spec}(T_a)$, that is, T_a is a field. □

5. Jacobson schemes

5.1. *Definition.* A map $f : X \to Y$ of topological spaces is a *quasi-homeomorphism* if $U \mapsto f^{-1}(U)$ is a bijection from the open subsets of Y to the open subsets of X (or equivalently, from the closed subsets of Y to the closed subsets of X).

Any quasi-homeomorphism f is, in particular, continuous.

If f is an injective quasi-homeomorphism, then f is a homeomorphism of X onto its image $Y' = f(X) \subseteq Y$, and Y' has the property that $Z \cap Y'$ is dense in Z, for every closed $Z \subseteq Y$. Conversely, this property of a subspace $Y' \subseteq Y$ implies that the inclusion map is a quasi-homeomorphism.

A bijective quasi-homeomorphism is a homeomorphism.

If $f : X \to Y$ is a quasi-homeomorphism, then the functor f_* defines an equivalence of categories from presheaves on X to presheaves on Y, and likewise for sheaves. In other words, quasi-homeomorphic spaces effectively have identical sheaf theories. For sheaves, the equivalence is given in the opposite direction by the inverse image functor f^{-1}, since f^{-1} is left adjoint to f_*.

5.2. *Definition.* A topological space X is *Jacobson* if every closed subset $Z \subseteq X$ is equal to the closure of the set Z_{cl} of closed points in Z. Note that $Z_{\text{cl}} = Z \cap X_{\text{cl}}$, since Z is closed.

5.3. *Lemma.* The following conditions are equivalent:

(i) X is Jacobson.

(ii) The inclusion $X_{\text{cl}} \hookrightarrow X$ is a quasi-homeomorphism.

(iii) Every non-empty locally closed subset of X contains a point of X_{cl}.

The proof is an easy exercise.

5.4. *Lemma.* (i) Every closed subset of a Jacobson space is Jacobson.

(ii) Every open subset of a Jacobson space is Jacobson.

(iii) If a space X has an open cover $X = \bigcup_\alpha X_\alpha$, where each X_α is Jacobson, then X is Jacobson.

Proof. Part (i) is immediate from the definition.

For (ii), let X be Jacobson, $U \subseteq X$ open, $W \subseteq U$ locally closed, $W \neq \emptyset$. Then W is also locally closed in X, hence contains a point $p \in X_{\mathrm{cl}}$. But p is then also a closed point of U. Using Lemma 5.3, (iii), this shows that U is Jacobson.

For (iii), let $W \subseteq X$ be locally closed and non-empty. Then $W \cap X_\alpha$ is locally closed and non-empty for some α, hence $W \cap X_\alpha$ contains a closed point p of X_α. We claim that p is a closed point of X (which is not obvious, since X_α need not be closed). We have $\{p\} = \overline{\{p\}} \cap X_\alpha$, so $\{p\}$ is locally closed. For every β such that $p \in X_\beta$, it follows that p is closed in X_β, since the locally closed subset $\{p\}$ must contain a closed point of X_β. Now suppose $q \in \overline{\{p\}}$. For some β, we have $q \in X_\beta$. Since $q \in \overline{\{p\}}$ and X_β is open, we also have $p \in X_\beta$. But then q belongs to the closure of $\{p\}$ in X_β, so $q = p$. Again using Lemma 5.3, (iii), this shows that X is Jacobson. □

5.5. For any commutative ring R, the closed points of $\mathrm{Spec}(R)$ are the maximal ideals $\mathfrak{m} \subseteq R$. The intersection of all maximal ideals $\mathfrak{m}$ containing a given ideal $I \subseteq R$ is called the *Jacobson radical* $\mathrm{rad}(I)$ of I. Clearly $\mathrm{rad}(I)$ is the unique radical ideal J such that $V(J) = \overline{Z_{\mathrm{cl}}}$, where $Z = V(I)$. Hence $\mathrm{Spec}(R)$ is Jacobson if and only if R has the property that $\mathrm{rad}(I) = \sqrt{I}$ for every ideal $I \subseteq R$. A ring R satisfying this condition is a *Jacobson ring.*

5.6. *Proposition.* Let X be a scheme. The following conditions are equivalent.

(i) X is Jacobson (meaning that its underlying topological space is Jacobson).

(ii) For every open affine $U = \mathrm{Spec}(R) \subseteq X$, the ring R is Jacobson.

(iii) There exists a covering $X = \bigcup_\alpha U_\alpha$ of X by open affines $U_\alpha = \mathrm{Spec}(R_\alpha)$ such that R_α is Jacobson.

Proof. Immediate from Lemma 5.4, (ii) and (iii).

5.7. *Lemma.* Let $X = \mathrm{Spec}(R)$ be an affine scheme. Suppose that for every $f \in R$ and $z \in X_f$, if z is closed in X_f, then z is closed in X. Then X is Jacobson.

Proof. Since the open sets X_f form a base of the topology, every non-empty locally closed subset $W \subseteq X$ contains a non-empty locally closed subset of the form $Z = V(I) \cap X_f$. It suffices to show that every such Z contains a closed point of X.

Since Z is the underlying space of an affine scheme, namely $\mathrm{Spec}(R_f/I_f)$, it has at least one closed point z (because a non-zero ring always has at least one maximal ideal). Since Z is closed in X_f, z is also a closed point of X_f. By hypothesis, z is then a closed point of X. □

We remark that the property of affine schemes Z used in the proof, that if $Z \neq \emptyset$, then Z has a closed point, does not hold for every topological space nor even for every scheme, although it does hold for every affine, quasi-compact, Jacobson, or locally Noetherian scheme.

6. A theorem on Jacobson schemes

6.1. *Theorem.* Let $f\colon X \to Y$ be a morphism of schemes locally of finite type, where Y is Jacobson. Then:

(i) X is Jacobson.

(ii) $f(X_{\mathrm{cl}}) \subseteq Y_{\mathrm{cl}}$.

(iii) For every $p \in X_{\mathrm{cl}}$, with $q = f(p)$, the homomorphism of residue fields $\mathbf{k}_q \hookrightarrow \mathbf{k}_p$ induced by f is a finite algebraic extension.

6.2. In the rest of this section we give the proof of Theorem 6.1. The first step is to observe that the local nature of the hypothesis and the conclusions allows us to reduce to the case where X and Y are affine. This requires a bit of care regarding conclusion (ii), because it is not generally true that a closed point q of an open subset W of Y is a closed point of Y. However, this does hold given that Y is Jacobson, since the set $\overline{\{q\}} \cap W = \{q\}$ is locally closed, hence contains a closed point of Y.

6.3. We now assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. By Proposition 2.5, B is a finitely generated A-algebra, that is, $B = A[x_1, \ldots, x_n]/I$. We can factor the morphism $X \to Y$ into a chain of morphisms

$$X \to Y_n \to \cdots \to Y_1 \to Y_0 = Y,$$

where $Y_i = \mathrm{Spec}(A[x_1, \ldots, x_i])$. Assuming that Theorem 6.1 holds for each step in the chain, we can conclude first that all the Y_i and X are Jacobson, and then that the other conclusions of the theorem also hold for each morphism $Y_i \to Y$ and for $X \to Y$.

The conclusions of Theorem 6.1 are trivial for the morphism $X \to Y_n$, which is a closed embedding. We are left to verify Theorem 6.1 for the morphisms $Y_{i+1} \to Y_i$, that is, for morphisms of the form $\pi \colon \mathbb{A}^1(R) \to \mathrm{Spec}(R)$.

6.4. Now suppose that $Y = \mathrm{Spec}(R)$, $X = \mathbb{A}^1_R$, and $f \colon X \to Y$ is the structure morphism $\pi \colon \mathbb{A}^1_R \to \mathrm{Spec}(R)$ of $\mathbb{A}^1_R$ as a scheme over R.

We first prove (ii) and (iii). By Proposition 4.3, if $p \in X_{\mathrm{cl}}$ is a closed point, then $q = \pi(p)$ is a locally closed point of Y. Since Y is Jacobson, the locally closed subset $\{q\}$ contains a closed point of Y. In other words, q is a closed point, giving (ii). Proposition 4.2(ii) gives (iii).

For (i), we are to prove that $R[x]$ is Jacobson. By Lemma 5.7, it suffices to show that if $f(x) \in R[x]$, and p is a closed point of $(\mathbb{A}^1_R)_f$, then p is closed in $\mathbb{A}^1_R$. Our hypothesis on p implies that it is a locally closed point of $\mathbb{A}^1_R$. Proposition 4.3 then shows that $q = \pi(p)$ is a locally closed point of $\mathrm{Spec}(R)$. As before, since R is Jacobson, q is in fact a closed point of $\mathrm{Spec}(R)$. By Proposition 4.2(i), p is a closed point of the closed subset $\pi^{-1}(q)$, hence a closed point of $\mathbb{A}^1_R$.

7. The equivalence between varieties and schemes, Part I

We are now ready to establish the affine case of the equivalence in §0.2, and also to construct the functor that will provide the general case of the equivalence in the direction from reduced algebraic k-schemes to classical varieties.

7.1. If X is a topological space, let $\mathcal{F}_X$ denote the sheaf of all k-valued functions on open subsets of X. In particular, $\mathcal{F}_X$ is a sheaf of k-algebras. If $\mathcal{O}_X \subseteq \mathcal{F}_X$ is a subsheaf of k-algebras, let us call the pair $(X, \mathcal{O}_X)$ a *function ringed space over* k. We define a morphism of function ringed spaces over k in the same way that we defined a morphism of classical algebraic varieties in 1.12. Thus the algebraic varieties form a full subcategory of the function ringed spaces over k.

We begin by constructing a functor Cl(−) from reduced algebraic k-schemes Y to function ringed spaces X over k, which we will subsequently see gives the equivalence with classical algebraic varieties.

7.2. Let Y be an algebraic k-scheme, where k is an algebraically closed field. By Theorem 6.1(iii), every closed point $p \in Y$ has has residue field $\mathbf{k}_p = k$, that is, p is the image of a k-point of Y. Conversely, the image p of every k-point $\operatorname{Spec}(k) \to Y$ is closed, since if $U = \operatorname{Spec}(R)$ is any affine open neighborhood of p, then p is given by the kernel of a k-algebra homomorphism $\varphi\colon R \to k$, and φ is necessarily surjective, so $\ker(\varphi)$ is a maximal ideal. Hence the set of closed points Y_{cl} is equal to the image of $Y(k)$ in Y.

Let $X = Y_{\mathrm{cl}}$, with the subspace topology in Y, and let $i\colon X \hookrightarrow Y$ be the inclusion map. Let $\mathcal{F}_X$ be the sheaf of all k-valued functions on open subsets of X.

There is an evaluation homomorphism $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{F}_X$ sending $s \in \mathcal{O}_Y(U)$ to the function $i^\flat(s) \in \mathcal{F}_X(X \cap U)$ whose value at a closed point $p \in U$ is the image of the germ $s_p \in \mathcal{O}_{Y,p}$ in the residue field $\mathbf{k}_p = \mathcal{O}_{Y,p}/\mathfrak{m}_p = k$. Let $\mathcal{O}_X \subseteq \mathcal{F}_X$ be the subsheaf whose sections are the functions given locally by evaluating sections of $\mathcal{O}_Y$ in this way. In other words, $\mathcal{O}_X$ is the image of the sheaf homomorphism $i^\sharp\colon i^{-1}\mathcal{O}_Y \to \mathcal{F}$ corresponding to $i^\flat$ via the adjoint functors i^{-1} and i_*. By Theorem 6.1(i), Y is Jacobson and i is a quasi-homeomorphism, so i^{-1} and i_* give an equivalence between sheaves on X and sheaves on Y. Hence we can also describe $\mathcal{O}_X$ as the unique subsheaf of $\mathcal{F}_X$ such that $i_*\mathcal{O}_X$ is the image of $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{F}_X$ (if i were not a quasi-homeomorphism, the subsheaf $\operatorname{im}(i^\flat) \subseteq i_*\mathcal{F}_X$ would not necessarily be of the form $i_*\mathcal{O}$ for some $\mathcal{O} \subseteq \mathcal{F}_X$).

We define Cl(Y) to be the function ringed space $(X, \mathcal{O}_X)$ over k. By Theorem 6.1(iii) and Proposition 2.4, every k-morphism $Y \to Y'$ of algebraic k-schemes carries $X = Y_{\mathrm{cl}}$ into $X' = Y'_{\mathrm{cl}}$. This given, the construction of $\mathcal{O}_X$ from $\mathcal{O}_Y$ is clearly functorial, making Cl(−) a functor from algebraic k-schemes to function ringed spaces over k.

To construct the functor Cl(−) we do not actually need to assume that Y is a *reduced* algebraic k-scheme, but we do need this for the next result.

7.3. *Lemma.* Let Y be a reduced algebraic k-scheme, let $X = \mathrm{Cl}(Y)$ be the associated function ringed space over k, with inclusion map $i\colon X = Y_{\mathrm{cl}} \hookrightarrow Y$, and let $i^\flat\colon \mathcal{O}_Y \to i_*\mathcal{F}_X$ be the sheaf homomorphism in the construction of $\mathcal{O}_X$. Then $i^\flat$ induces an isomorphism $i^\flat\colon \mathcal{O}_Y \cong i_*\mathcal{O}_X$.

Proof. By construction, $i_*\mathcal{O}_X$ is the image of $i^\flat$, so we are to show that $i^\flat$ is injective. It suffices to show that $i^\flat_U\colon \mathcal{O}_Y(U) \to \mathcal{F}_X(X \cap U)$ is injective for all affine open subsets $U = \operatorname{Spec}(R)$ of Y, since they form a base of the topology. We have $R = \mathcal{O}_Y(U)$, and $X \cap U$ is the set of maximal ideals in R. Hence the kernel of $i^\flat_U$ is the intersection of all maximal ideals in R. But since Y is reduced and Jacobson, R is a reduced Jacobson ring, so the intersection of its maximal ideals is $\sqrt{0} = 0$. □

7.4. *Proposition.* Let X be a classical affine variety with coordinate ring $R(X)$ and sheaf of regular functions $\mathcal{O}_X$. The function ringed space Cl(Y) associated to the reduced algebraic k-scheme $Y = \operatorname{Spec}(R(X))$ is isomorphic to $(X, \mathcal{O}_X)$.

Proof. We essentially proved this already in §1. To be precise, we saw in §1.6 that X is canonically homeomorphic to the image of $Y(k)$ in Y, which we now know to be Y_{cl}, and in §1.11 we characterized $\mathcal{O}_X$ as the image of the sheaf homomorphism $i^\sharp\colon i^{-1}\mathcal{O}_Y \to \mathcal{F}_X$. □

7.5. *Corollary.* Let X be a classical affine variety. Every global regular function on X is a polynomial function. In other words, the containment $R(X) \subseteq \mathcal{O}_X(X)$ is an equality.

Proof. Let $Y = \mathrm{Spec}(R(X))$. Then $\mathcal{O}_Y(Y) = R(X)$, and Proposition 7.4 and Lemma 7.3 give $\mathcal{O}_Y(Y) = \mathcal{O}_X(X)$. □

7.6. *Corollary.* Every morphism between classical affine varieties over k is given by a polynomial map.

Proof. If $\varphi\colon X \to X'$ is a morphism and $g \in \mathcal{O}_{X'}(X')$ then $g\circ\varphi \in \mathcal{O}_X(X)$. By Corollary 7.5, it follows that if $g \in R(X')$, then $\varphi \in R(X)$. This condition is the definition of a polynomial map. □

7.7. *Lemma.* Let k be an algebraically closed field. Every radical ideal $I = \sqrt{I}$ in the polynomial ring $k[x_1,\ldots,x_n]$ is the ideal $\mathcal{I}(X)$ of an affine variety $X \subseteq k^n$.

Proof. By Theorem 6.1(i) applied to $\mathrm{Spec}(k[\mathbf{x}]) \to \mathrm{Spec}(k)$, the polynomial ring $k[\mathbf{x}]$ is Jacobson. By Theorem 6.1(iii), since k is algebraically closed, every maximal ideal $\mathfrak{m} \subseteq k[\mathbf{x}]$ has residue field $k[\mathbf{x}]/\mathfrak{m} = k$. In other words, by §1.6, $\mathfrak{m}$ is the kernel of the evaluation homomorphism $\mathrm{ev}_a\colon k[\mathbf{x}] \to k$ for some point $a \in k^n$.

Since R is Jacobson, $I = \sqrt{I}$ is equal to the intersection of the maximal ideals $\mathfrak{m}$ containing I. But these are exactly the maximal ideals corresponding to points $a \in X = V(I)$, so their intersection is $\mathcal{I}(X)$. □

Remark. Lemma 7.7 is Hilbert's Nullstellensatz. From the proof we see that the Nullstellensatz is a corollary to Theorem 6.1.

7.8. *Corollary.* If k is algebraically closed, then every finitely generated reduced k-algebra is isomorphic to the coordinate ring $R(X)$ of a classical affine variety X over k.

7.9. *Corollary.* If Y is a reduced algebraic scheme over an algebraically closed field k, then the function ringed space $X = \mathrm{Cl}(Y)$ is a classical algebraic variety over k. If Y is affine, then so is X.

Proof. Proposition 7.4 and Corollary 7.8 imply that if Y is a reduced algebraic affine k-scheme, then $\mathrm{Cl}(Y)$ is isomorphic to a classical affine variety, namely, any variety X for which $Y = \mathrm{Spec}(R(X))$.

In the general case, it follows that for every affine open subset $U \subseteq Y$, the open set $X \cap U$ in $X = \mathrm{Cl}(Y)$ is isomorphic to a classical affine variety. Since X is covered by open sets of this form, X is a classical algebraic variety. □

7.10. *Proposition.* The functor $\mathrm{Cl}(-)$ restricts to an equivalence of categories from reduced algebraic affine k-schemes to classical affine varieties over k.

Proof. By Corollary 7.9, the functor $\mathrm{Cl}(-)$ sends reduced algebraic affine k-schemes to classical affine varieties over k. By Corollary 7.6, we can identify the morphisms in the category of classical affine varieties with polynomial maps.

We had already seen in §1.8 that polynomial maps $X \to X'$ correspond bijectively to k-algebra homomorphisms $R(X') \to R(X)$ and thus to k-morphisms of schemes from $Y = \mathrm{Spec}(R(X))$ to $Y' = \mathrm{Spec}(R(X'))$. Hence the constructions in §1.8 give equivalences between the categories of (i) classical affine varieties over k, (ii) finitely-generated reduced k-algebras (with arrows reversed), and (iii) reduced algebraic affine schemes over k. It is straightforward to see from the definitions that the functor which gives the correspondence from the last of these to the first coincides (up to canonical functorial isomorphism) with the functor $\mathrm{Cl}(-)$. □

7.11. *Corollary.* Open affine subvarieties U form a base of the topology on any classical variety X.

Proof. If X is affine, say $X = \mathrm{Cl}(Y)$, where $Y = \mathrm{Spec}(R(X))$, then the open subsets $X_f = \mathrm{Cl}(Y_f)$ form a base of the topology, and they are affine by Proposition 7.10. The general case follows from the affine case, since X can be covered by affine open subsets, by definition.

We have described the functor $\mathrm{Cl}(-)$ that gives the equivalence in §0.2 in the direction from schemes to varieties, and established the affine case of the equivalence. In order to extend this to the general case, we first need a coordinate-free construction of the functor inverse to $\mathrm{Cl}(-)$, not dependent upon the choice of an affine covering. The inverse functor will be given by 'soberization.' We develop the essential notions in the next section. In §9, we then state and prove the full equivalence.

8. Sober spaces

What kind of topological space X can be the underlying space of a scheme? In general, such spaces satisfy only the weakest of the standard separation axioms (X is T_0), but they have another property: X is a *sober space*, which means that every irreducible closed subset of X is the closure of a unique point.

In this section we discuss sober spaces and the soberization functor, and prove that schemes are sober. Aside from their utility in completing the proof of the equivalence in §0.2, these concepts can be helpful for developing some intuitive understanding of the topology of schemes in general.

8.1. A topological space Z is *irreducible* if Z is non-empty, and Z is not a union of two proper closed subsets (hence not a union of any finite number of proper closed subsets).

Other equivalent ways to formulate the condition that a non-empty space Z is irreducible are (a) every intersection $U_1 \cap U_2$ of two non-empty open subsets of Z is non-empty; or (b) every non-empty open subset of Z is dense in Z.

8.2. *Proposition.*

(i) If $f\colon Z \to X$ is continuous and Z is irreducible, then $f(Z)$ is irreducible.

(ii) If $Z \subseteq X$ is an irreducible subspace, then the closure $\overline{Z}$ is irreducible.

(iii) Every non-empty open subset of an irreducible space is irreducible.

The proof is easy. Note, in particular, that the closure $\overline{\{x\}}$ of any point in any space X is always irreducible.

8.3. *Definition.* A topological space X is *sober* if every irreducible closed subset Z of X is the closure $Z = \overline{\{z\}}$ of a unique point z.

As a trivial example, any Hausdorff space is sober, since its only irreducible subspaces are the one-point sets $\{x\}$. More importantly, as we will see next, every scheme is sober.

Recall that, by definition, a space X is T_0 if for every two distinct points of X, there is an open subset of X that contains exactly one of them. An equivalent condition is that $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$ for all $x, y \in X$. In particular, every sober space is T_0.

8.4. *Lemma.* If X has a covering by sober open subspaces U_α, then X is sober.

Proof. It is easy to see that the open covering by T_0 spaces U_α implies that X is T_0.

Let $Z \subseteq X$ be an irreducible closed subset. For each α such that $Z \cap U_\alpha \neq \emptyset$, we have $Z \cap U_\alpha = \overline{\{z_\alpha\}} \cap U_\alpha$ for a unique $z_\alpha \in U_\alpha$, since U_α is sober. If $Z \cap U_\alpha \neq \emptyset$ and $Z \cap U_\beta \neq \emptyset$, then $Z \cap U_\alpha \cap U_\beta \neq \emptyset$, since Z is irreducible. Since $\{z_\alpha\}$ is dense in $Z \cap U_\alpha$, it follows that $z_\alpha \in Z \cap U_\beta \subseteq \overline{\{z_\beta\}}$. By symmetry, we also have $z_\beta \in \overline{\{z_\alpha\}}$, hence $\overline{\{z_\alpha\}} = \overline{\{z_\beta\}}$. Since X is T_0, $z_\alpha = z_\beta$. This shows that there is a single point $z \in Z$ such that

$$Z \cap U_\alpha = \overline{\{z\}} \cap U_\alpha$$

for all α such that $Z \cap U_\alpha \neq \emptyset$. The same identity holds trivially if $Z \cap U_\alpha = \emptyset$, since the right hand side is a subset of the left hand side. Since (U_α) is an open cover of X, it follows that $Z = \overline{\{z\}}$, and since X is T_0, z is unique. $\square$

8.5. *Proposition.* Every scheme is sober.

Proof. By Lemma 8.4, we can reduce to the case of an affine scheme $X = \operatorname{Spec}(R)$. The irreducible closed subsets of X are then the sets $Z = V(P)$, where P is a prime ideal in R. The closure of any subset $Y \subseteq \operatorname{Spec}(R)$ is given by $\overline{Y} = V(I)$, where $I = \cap_{Q \in Y} Q$. In particular, $\overline{\{P\}} = V(P) = Z$. If we also had $\overline{\{P'\}} = Z$, then we would have $V(P) = V(P')$, hence $P \subseteq P'$ and $P' \subseteq P$, so P is unique. $\square$

8.6. Given any topological space X, let $\operatorname{Sob}(X)$ denote the set of irreducible closed subsets of X. For every closed $Y \subseteq X$, define

$$V(Y) = \{Z \in \operatorname{Sob}(X) \mid Z \subseteq Y\}.$$

The sets $V(Y)$ are the closed subsets of a topology on $\operatorname{Sob}(X)$, by virtue of the identities

$$V(\emptyset) = \emptyset, \quad V(X) = \operatorname{Sob}(X), \quad V(\bigcap_\alpha Y_\alpha) = \bigcap_\alpha V(Y_\alpha), \quad V(Y_1 \cup Y_2) = V(Y_1) \cup V(Y_2).$$

The first of these holds because an irreducible space is non-empty by definition. For the last, if $Z \in V(Y_1 \cup Y_2)$ then, since Z is irreducible, $Z \in V(Y_1)$ or $Z \in V(Y_2)$. Thus

$V(Y_1 \cup Y_2) \subseteq V(Y_1) \cup V(Y_2)$, and the opposite containment is trivial. The remaining two identities above are trivial as well.

There is a canonical map

$$i\colon X \to \mathrm{Sob}(X)$$

defined by $i(x) = \overline{\{x\}}$. One checks immediately that $i^{-1}(V(Y)) = Y$ for every closed $Y \subseteq X$. The correspondence $Y \mapsto V(Y)$ from closed subsets of X to closed subsets of $\mathrm{Sob}(X)$ is surjective by definition. The identity $i^{-1}(V(Y)) = Y$ then implies that $V(-)$ and i^{-1} are inverse bijections between the closed subsets of X and $\mathrm{Sob}(X)$. In particular, we have the following.

8.7. *Proposition.* For any space X, the canonical map $i\colon X \to \mathrm{Sob}(X)$ is a quasi-homeomorphism.

8.8. *Proposition.* For any space X, $\mathrm{Sob}(X)$ is a sober space.

Proof. Since i is a quasi-homeomorphism, i^{-1} induces a bijection from irreducible closed subsets of $\mathrm{Sob}(X)$ to irreducible closed subsets of X, with inverse given by $V(-)$. Thus every irreducible closed subset of $\mathrm{Sob}(X)$ is $V(Z)$ for a unique irreducible closed $Z \subseteq X$. The smallest closed $Y \subseteq X$ such that $Z \in V(Y)$, that is, such that $Z \subseteq Y$, is clearly $Y = Z$, so $V(Z)$ is the smallest closed subset of $\mathrm{Sob}(X)$ containing Z, that is, $V(Z) = \overline{\{Z\}}$ in $\mathrm{Sob}(X)$. For uniqueness, if $\overline{\{Z\}} = \overline{\{Z'\}}$, then $Z \in V(Z')$ and $Z' \in V(Z)$ imply $Z \subseteq Z' \subseteq Z$, so $Z = Z'$. □

8.9. *Lemma.* If X is sober, then the canonical map $i\colon X \to \mathrm{Sob}(X)$ is a homeomorphism.

Proof. By Proposition 8.8, i is a quasi-homeomorphism. The assertion that i is a bijection is equivalent to the definition of the space X being sober. □

8.10. Given a continuous map $f\colon X \to X'$, we define a map $\mathrm{Sob}(f)\colon \mathrm{Sob}(X) \to \mathrm{Sob}(X')$ by

$$\mathrm{Sob}(f)(Z) = \overline{f(Z)}.$$

This makes sense by Proposition 8.2, (i) and (ii). For $Y' \subseteq X'$ closed, we have $\mathrm{Sob}(f)(Z) \in V(Y') \Leftrightarrow f(Z) \subseteq Y' \Leftrightarrow Z \subseteq f^{-1}(Y')$. In other words, $\mathrm{Sob}(f)^{-1}(V(Y')) = V(f^{-1}(Y'))$. Hence $\mathrm{Sob}(f)$ is continuous. It is easy to see that $\mathrm{Sob}(g \circ f) = \mathrm{Sob}(g) \circ \mathrm{Sob}(f)$ and that $\mathrm{Sob}(1_X)$ is the identity map on $\mathrm{Sob}(X)$, so

$$\mathrm{Sob}(-)\colon \underline{\mathrm{Top}} \to \text{(sober spaces)}$$

is a functor.

8.11. For any continuous map $f\colon X \to X'$, one checks immediately that the diagram

$$\begin{array}{ccc} \mathrm{Sob}(X) & \xrightarrow[\mathrm{Sob}(f)]{} & \mathrm{Sob}(X') \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

commutes. In other words, the canonical maps $i_X\colon X \to \mathrm{Sob}(X)$ give a functorial map from the identity functor $\mathrm{id}_{\underline{\mathrm{Top}}}$ to $j \circ \mathrm{Sob}$, where $j\colon \text{(sober spaces)} \to \underline{\mathrm{Top}}$ is the inclusion functor.

8.12. Recall that, by definition, a topological space X is T_1 if every point of X is closed. If X is T_1, then the singleton sets $\{p\}$ are the minimal irreducible closed subsets of X, and are therefore the closed points of $\mathrm{Sob}(X)$. In other words, $i\colon X \to \mathrm{Sob}(X)$ is injective with image $i(X) = \mathrm{Sob}(X)_{\mathrm{cl}}$. By Proposition 8.7 and Lemma 5.3, (ii), it follows that $\mathrm{Sob}(X)$ is Jacobson, and i is a homeomorphism of X onto $\mathrm{Sob}(X)_{\mathrm{cl}}$. If $f\colon X \to X'$ is a continuous map of T_1 spaces, we also see that $\mathrm{Sob}(f)$ carries $\mathrm{Sob}(X)_{\mathrm{cl}}$ into $\mathrm{Sob}(X')_{\mathrm{cl}}$.

Conversely, if Y is a Jacobson sober space, then $j\colon Y_{\mathrm{cl}} \hookrightarrow Y$ is a quasi-homeomorphism, and therefore the induced map $\mathrm{Sob}(Y_{\mathrm{cl}}) \to Y$ is a homeomorphism.

Let $\underline{\mathrm{JacSob}}$ denote the category of Jacobson sober spaces and continuous maps $f\colon Y \to Y'$ such that $f(Y_{\mathrm{cl}}) \subseteq Y'_{\mathrm{cl}}$. The preceding observations prove the following result.

Proposition. The functor $\mathrm{Sob}(-)$ restricts to an equivalence of categories

$$\mathrm{Sob}(-) : (T_1 \text{ spaces}) \to \underline{\mathrm{JacSob}}$$

with inverse $Y \mapsto Y_{\mathrm{cl}}$. □

8.13. To state and prove the full equivalence between varieties and schemes (Theorem 9.1) we will only use the facts about sober spaces and the functor $\mathrm{Sob}(-)$ developed in §§8.1–8.12, above. For completeness, however, we conclude this section with some additional results on the topological significance of sober spaces, independent of their relevance to schemes.

Proposition. The canonical map $i_X\colon X \to \mathrm{Sob}(X)$ has the universal property that every continuous map $f\colon X \to Y$, where Y is sober, factors uniquely through i_X. Equivalently, the functor $\mathrm{Sob}(-)$ is left adjoint to the inclusion $j\colon$ (sober spaces) $\to \underline{\mathrm{Top}}$, with i_X giving the unit of the adjunction.

Proof. The equivalence of the two statements is easy and purely category-theoretic.

For the universal property, suppose $f\colon X \to Y$ is continuous and Y is sober. Then i_Y is a homeomorphism, by Lemma 8.9. Taking $X' = Y$ in diagram (8.11), we see that $f = g \circ i_X$, where $g = i_Y^{-1} \circ \mathrm{Sob}(f)$. Thus f factors through i_X.

To show that g is unique, suppose $h\colon \mathrm{Sob}(X) \to Y$ is another continuous map such that $f = h \circ i_X$. Then for any closed subset $T \subseteq Y$, we have $i_X^{-1}(g^{-1}(T)) = i_X^{-1}(h^{-1}(T)) = f^{-1}(T)$. Since i_X is a quasi-homeomorphism, this implies that $g^{-1}(T) = h^{-1}(T)$. Given any $Z \in \mathrm{Sob}(X)$, it follows that $g(Z)$ and $h(Z)$ belong to exactly the same closed subsets of Y. Since Y is T_0, this shows that $g = h$. □

Remark. More explicitly, given a point $Z \in \mathrm{Sob}(X)$, that is, an irreducible closed subset $Z \subseteq X$, the formula $g = i_Y^{-1} \circ \mathrm{Sob}(f)$ means that $g(Z)$ is the unique point y in the sober space Y such that $\overline{\{y\}} = \overline{f(Z)}$.

8.14. Let $f\colon X \to Y$ be a quasi-homeomorphism between arbitrary topological spaces. Then $W \mapsto f^{-1}(W)$ is a bijection from the irreducible closed subsets of Y to those of X, and its inverse is easily seen to be $Z \mapsto \overline{f(Z)}$, that is, $\mathrm{Sob}(f)$. By Lemma 8.9, $\mathrm{Sob}(f)$ is then a bijective quasi-homeomorphism, hence a homeomorphism.

In other words, the functor $\mathrm{Sob}\colon \underline{\mathrm{Top}} \to$ (sober spaces) factors through a functor $S\colon Q^{-1}\,\underline{\mathrm{Top}} \to$ (sober spaces), where $Q^{-1}\,\underline{\mathrm{Top}}$ is the category obtained by formally inverting all quasi-homeomorphisms in $\underline{\mathrm{Top}}$.

Composing the canonical functor $\underline{\text{Top}} \to Q^{-1}\,\underline{\text{Top}}$ with the inclusion of sober spaces into $\underline{\text{Top}}$ gives a functor $j\colon$ (sober spaces) $\to Q^{-1}\,\underline{\text{Top}}$. Using Proposition 8.7 and Lemma 8.9, one can verify that S and j are inverse up to functorial isomorphism, giving an equivalence of categories

$$Q^{-1}\,\underline{\text{Top}} \cong \text{(sober spaces)}.$$

In particular, the category $Q^{-1}\,\underline{\text{Top}}$ (which is a priori a *large category*, that is, the class of morphisms between two objects need not be a set) is an ordinary category, for which the category of sober spaces serves as a concrete natural model.

9. The equivalence between varieties and schemes, Part II

9.1. *Theorem.* Let k be an algebraically closed field. The category of reduced algebraic k-schemes is equivalent to the category of classical algebraic varieties over k.

In one direction, the equivalence is given by the functor $\text{Cl}(-)$ in §7.2, which sends an algebraic k-scheme Y to its space of closed points $X = Y_{\text{cl}}$ (which is also the image of $Y(k)$ in Y), with $\mathcal{O}_X$ the sheaf of rings of k-valued functions constructed by evaluating sections of $\mathcal{O}_Y$ as functions on open subsets of X.

In the other direction, it is given by the functor sending a classical variety X to the ringed space $(Y, \mathcal{O}_Y) = (\text{Sob}(X), i_*\mathcal{O}_X)$, where $i\colon X \to \text{Sob}(X)$ is the canonical functorial map in §8.6.

The variety X is affine if and only if the corresponding scheme Y is affine. In this case, we have $Y = \text{Spec}(R(X))$, where $R(X)$ is the coordinate ring of X.

9.2. Before proving Theorem 9.1, we first explain how the correspondence $X \mapsto (\text{Sob}(X), i_*\mathcal{O}_X)$ is a functor; that is, what it does to a morphism of function ringed spaces $f\colon X \to X'$.

The map of underlying spaces is of course just $\text{Sob}(f)\colon \text{Sob}(X) \to \text{Sob}(X')$. To get a morphism of ringed spaces (ringed in k-algebras) from $(\text{Sob}(X), i_*\mathcal{O}_X)$ to $(\text{Sob}(X'), i'_*\mathcal{O}_{X'})$, we need to specify in addition a homomorphism of sheaves of k-algebras $f^\flat\colon i'_*\mathcal{O}_{X'} \to (\text{Sob}(f))_* i_*\mathcal{O}_X$. By §8.11, we have $\text{Sob}(f) \circ i = i' \circ f$, so $(\text{Sob}(f))_* i_*\mathcal{O}_X = i'_* f_*\mathcal{O}_X$. By the definition of a morphism of function ringed spaces, the canonical homomorphism $f^\circ\colon \mathcal{F}_{X'} \to f_*\mathcal{F}_X$ given by composing functions with f induces a homomorphism of subsheaves $f^\circ\colon \mathcal{O}_{X'} \to f_*\mathcal{O}_X$. The desired homomorphism $f^\flat\colon i'_*\mathcal{O}_{X'} \to i'_* f_*\mathcal{O}_X$ is then $f^\flat = i'_*(f^\circ)$. It is a straightforward exercise to check that this construction is compatible with composition of morphisms.

9.3. We now prove Theorem 9.1. We have already done most of the work and have only to assemble the pieces. Let us (improperly) use the same notation $\text{Sob}(-)$ for the functor $X \mapsto (\text{Sob}(X), i_*\mathcal{O}_X)$ on function ringed spaces that we use for the soberization functor on topological spaces.

For the affine case, we know by Proposition 7.10 that $\text{Cl}(-)$ gives an equivalence from reduced algebraic affine k-schemes to classical affine varieties, and that $X = \text{Cl}(Y)$ corresponds to $Y = \text{Spec}(R(X))$. We want to show that $Y \cong \text{Sob}(X)$. Since $X = Y_{\text{cl}}$, and Y is Jacobson (Theorem 6.1) and sober (Proposition 8.5), Proposition 8.12 gives a functorial homeomorphism $Y \cong \text{Sob}(X)$. By Lemma 7.3, we have $\mathcal{O}_Y \cong i_*\mathcal{O}_X$, where $i\colon X = Y_{\text{cl}} \hookrightarrow Y$

is the inclusion map. Thus $(Y, \mathcal{O}_Y) \cong (\mathrm{Sob}(X), i_* \mathcal{O}_X)$, via a canonical isomorphism that is easily seen to be functorial.

For the general case, we must first verify that if Y is a reduced algebraic k-scheme, then $\mathrm{Cl}(Y)$ is a classical variety, and that if X is a classical variety, then $\mathrm{Sob}(X)$ is a reduced algebraic k-scheme. The first of these assertions is Corollary 7.9. For the second, given a classical variety X, we can cover it by open subsets U such that $(U, \mathcal{O}_X|U)$ is affine. The open sets $W = \mathrm{Sob}(U)$ cover $Y = \mathrm{Sob}(X)$, and $(W, (i_* \mathcal{O}_X)|W) = (\mathrm{Sob}(U), (i_U)_*(\mathcal{O}_X|U))$ is a reduced algebraic affine k-scheme by the affine case of the equivalence. Hence $(Y, \mathcal{O}_Y) = (\mathrm{Sob}(X), i_* \mathcal{O}_X)$ is a reduced algebraic k-scheme.

In §9.2, we explained how $\mathrm{Sob}(-)$ is a functor to ringed spaces. A morphism of schemes is defined to be a *local* morphism of locally ringed spaces. Thus we also need to check that if $f\colon X \to X'$ is a morphism of varieties, and $Y = \mathrm{Sob}(X)$, $Y' = \mathrm{Sob}(X')$ are the corresponding schemes, then the ringed space morphism $g = \mathrm{Sob}(f)\colon Y \to Y'$ is local. This too reduces to the affine case. For any $y \in Y$, we can choose an affine open neighborhood $V \subseteq Y'$ of $g(y)$, and an affine open neighborhood $U \subseteq g^{-1}(V)$ of y. Then $X \cap U$ and $X' \cap V$ are affine varieties, with $f(X \cap U) \subseteq X' \cap V$, and $g\colon U \to V$ is the morphism of affine schemes corresponding to $f\colon X \cap U \to X' \cap V$, hence it is local.

It remains to show that $\mathrm{Cl}(-)$ and $\mathrm{Sob}(-)$ are inverse to one another, up to functorial isomorphisms $X \cong \mathrm{Cl}(\mathrm{Sob}(X))$ and $Y \cong \mathrm{Sob}(\mathrm{Cl}(Y))$. As homeomorphisms on the underlying topological spaces, the required isomorphisms are given by Proposition 8.12. We need to promote these homeomorphisms to an isomorphism of ringed spaces $Y \cong \mathrm{Sob}(\mathrm{Cl}(Y))$ and an isomorphism of function ringed spaces $X \cong \mathrm{Cl}(\mathrm{Sob}(X))$.

Given either space X or Y, and constructing the other as $Y = \mathrm{Sob}(X)$ or $X = \mathrm{Cl}(Y)$, we have the inclusion map $i\colon X \hookrightarrow Y$. If X is given, then $\mathcal{O}_Y = i_* \mathcal{O}_X$ by definition. If Y is given, then we have a homomorphism $i^\flat\colon \mathcal{O}_Y \to i_* \mathcal{O}_X$ by the definition of $\mathcal{O}_X$. By Lemma 7.3, for every open affine subscheme $U \subseteq Y$, the restriction of $i^\flat$ to U is an isomorphism. Hence $i^\flat$ is an isomorphism.

Given Y, we can identify the space $\mathrm{Sob}(\mathrm{Cl}(Y))$ with Y. To promote $Y \cong \mathrm{Sob}(\mathrm{Cl}(Y))$ to a ringed space isomorphism, we must specify an isomorphism of sheaves of rings $\mathcal{O}_Y \cong i_* \mathcal{O}_X$, where $X = \mathrm{Cl}(Y)$. We take this isomorphism to be $i^\flat$. In principle we should now verify that the resulting ringed space isomorphism is functorial. This is more or less clear from the canonical nature of all constructions involved, so we omit the details.

Given X, we can identify the spaces X and $X' = \mathrm{Cl}(\mathrm{Sob}(X))$. We then have two sheaves of functions on X: the original sheaf $\mathcal{O}_X$, and a new sheaf $\mathcal{O}'_X$ coming from X'. We are to show that $\mathcal{O}_X = \mathcal{O}'_X$ as subsheaves of $\mathcal{F}_X$. Since open affine subsets form a base of the topology on X, by Corollary 7.11, the problem reduces to checking that $\mathcal{O}_X(U) = \mathcal{O}'_X(U)$ when U is affine. In this case, $\mathcal{O}_X(U)$ is the ring of polynomial functions $R(U)$ on U, by Corollary 7.8. Meanwhile, the open subset $W = \mathrm{Sob}(U)$ of $Y = \mathrm{Sob}(X)$ is the affine scheme $W = \mathrm{Spec}(R(U))$, and the map from the abstract k-algebra $\mathcal{O}'_X(U) = \mathcal{O}_W(W) = R(U)$ to functions on $U = X \cap W$ is just evaluation of elements of $R(U)$ as polynomial functions on U. Hence $\mathcal{O}_X(U) = \mathcal{O}'_X(U) = R(U)$. □