NOTES ON DERIVED CATEGORIES AND DERIVED FUNCTORS

MARK HAIMAN

References

You might want to consult these general references for more information:

1. R. Hartshorne, *Residues and Duality*, Springer Lecture Notes 20 (1966), is a standard reference.
2. C. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics 38 (1994), has a useful chapter at the end on derived categories and functors.
3. B. Keller, Derived categories and their uses, in *Handbook of Algebra, Vol. 1*, M. Hazewinkel, ed., Elsevier (1996), is another helpful synopsis.
4. J.-L. Verdier's thesis *Catégories dérivées* is the original reference; also his essay with the same title in *SGA 4-1/2*, Springer Lecture Notes 569 (1977).

The presentation here incorporates additional material from the following references:

5. P. Deligne, Cohomologie à supports propres, *SGA 4*, Springer Lecture Notes 305 (1973)
6. P. Deligne, Cohomologie à support propres et construction du foncteur $f^!$, appendix to *Residues and Duality* [1].
7. J.-L. Verdier, Base change for twisted inverse image of coherent sheaves, in *Algebraic Geometry, Bombay Colloquium 1968*, Oxford Univ. Press (1969)
8. N. Spaltenstein, Resolutions of unbounded complexes, *Compositio Math.* 65 (1988) 121–154
9. A. Neeman, Grothendieck duality via Bousfield's techniques and Brown representability, *J.A.M.S.* 9, no. 1 (1996) 205–236.

1. Basic concepts

Definition 1.1. An *additive category* is a category $\mathcal{A}$ in which $\mathrm{Hom}(A, B)$ is an abelian group for all objects A, B, composition of arrows is bilinear, and $\mathcal{A}$ has finite direct sums and a zero object. An *abelian category* is an additive category in which every arrow f has a kernel, cokernel, image and coimage, and the canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

The abelian categories of interest to us will be the category of modules over a ring (including the category of abelian groups, as $\mathbb{Z}$-modules), the category of sheaves of $\mathcal{O}_X$-modules on a ringed space X, and certain full abelian subcategories of these, such as the category

Math 256 Algebraic Geometry Fall 2013–Spring 2014.

of quasi-coherent sheaves of $\mathcal{O}_X$-modules. In these 'concrete' abelian categories, the objects have an underlying abelian group structure (on stalks, in the case of sheaves), and it is convenient to reason using elements of these groups. Reasoning based on elements can also be justified in abstract abelian categories by associating to an object A the abelian groups $\mathrm{Hom}(T, A)$ for various objects T. One thinks of arrows $T \to A$ as 'T-valued elements' of A, just as in the category of schemes we think of morphisms $T \to X$ as T-valued points of X.

Definition 1.2. A *complex* in an abelian category $\mathcal{A}$ is a sequence $A^\bullet$ of objects and maps (called *differentials*)

$$\cdots \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

such that $d^{i+1} \circ d^i = 0$ for all i. A homomorphism $f\colon A^\bullet \to B^\bullet$ of complexes consists of maps $f^i : A^i \to B^i$ commuting with the differentials. The complexes in $\mathcal{A}$ form an abelian category $\mathbf{C}(\mathcal{A})$. The object $H^i(A^\bullet) = \ker(d^i)/\operatorname{im}(d^{i-1})$ is the i-th *cohomology* of $A^\bullet$. I'll stick to cohomology indexing. It is also conventional sometimes to use "homology" indexing defined by $A_i = A^{-i}$, $H_i(A_\bullet) = H^{-i}(A^\bullet)$.

The *shift* $A[n]^\bullet$ of $A^\bullet$ is the complex with terms $A[n]^i = A^{i+n}$ and differentials $d^i_{A[n]} = (-1)^n d^{i+n}_A$. If $f\colon A^\bullet \to B^\bullet$ is a homomorphism of complexes, we define $f[n]\colon A[n]^\bullet \to B[n]^\bullet$ by $f[n]^i = f^{i+n}$. This makes the shift a functor from $\mathbf{C}(\mathcal{A})$ to itself.

The general principle governing sign rules is that all constructions involving complexes should be 'graded-commutative,' meaning that homogeneous operators s, t of degrees p, q should satisfy $ts = (-1)^{pq}\, st$. In the present case, the shift has degree n and the differentials have degree 1, so shift and differentials should commute up to a sign $(-1)^n$, which explains the definition $d^i_{A[n]} = (-1)^n d^{i+n}_A$. A homomorphism f, by contrast, has degree zero, so no sign appears in the definition $f[n]^i = f^{i+n}$.

An object A of $\mathcal{A}$ can be identified with the complex which is A in degree 0 and zero in all other degrees. This makes $\mathcal{A}$ a full subcategory of $\mathbf{C}(\mathcal{A})$. The shift $A[n]$ is then the complex which is A in degree $-n$.

Definition 1.3. A homomorphism of complexes $f\colon A^\bullet \to B^\bullet$ is a *quasi-isomorphism* ("qis" for short) if f induces isomorphisms $H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$ for all i. A complex is quasi-isomorphic to zero if and only if $H^i(A^\bullet) = 0$ for all i, that is, if $A^\bullet$ is *acyclic*.

Definition 1.4. The *mapping cone* of a homomorphism $f\colon A^\bullet \to B^\bullet$ is the complex $C(f)$ with terms $C(f)^n = A^{n+1} \oplus B^n$ (the same as $A[1] \oplus B$), and differentials $d^n(a, b) = (-d^{n+1}_A(a), f^{n+1}(a) + d^n_B(b))$.

Pictorially, $C(f)$ looks like this:

$$\begin{array}{ccccccc}
\to & A^0 & \xrightarrow{-d^0_A} & A^1 & \xrightarrow{-d^1_A} & A^2 & \to \\
\searrow & \oplus & \searrow & \oplus & \searrow & \oplus & \searrow \\
f^{-1} & & f^0 & & f^1 & & f^2 \\
\to & B^{-1} & \xrightarrow[d^{-1}_B]{} & B^0 & \xrightarrow[d^0_B]{} & B^1 & \to \\
 & (-1) & & (0) & & (1) &
\end{array},$$

displayed so that the columns are the terms of $C(f)$. You can verify immediately that it is in fact a complex.

Proposition 1.5. *(i) $C(f)$ is functorial in the triple $A \xrightarrow{f} B$.*

(ii) There is a canonical exact sequence of complexes

$$0 \to B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \to 0.$$

(iii) Let $K = \ker(f)$, $Q = \operatorname{coker}(f)$. The canonical maps $K \hookrightarrow A$ and $B \to Q$ factor through the maps i, p in (ii), as

$$K \xhookrightarrow{k} C(f)[-1] \xrightarrow{p[-1]} A, \quad B \xrightarrow{i} C(f) \xrightarrow{q} Q.$$

(iv) If f is injective, then $q\colon C(f) \underset{qis}{\xrightarrow{\simeq}} Q$ is a quasi-isomorphism.

(v) If f is surjective, then $k[1]\colon K[1] \underset{qis}{\xrightarrow{\simeq}} C(f)$ is a quasi-isomorphism.

(vi) If f is bijective, then $C(f)$ is acyclic.

(vii) For each i, the exact sequence in (ii) induces an exact sequence

$$H^i(B) \to H^i(C(f)) \to H^{i+1}(A).$$

Proof. Exercise. $\square$

Definition 1.6. A homomorphism of complexes $f\colon A^\bullet \to B^\bullet$ is *null-homotopic*, written $f \sim 0$, if there exist maps $s^i\colon A^i \to B^{i-1}$ such that $f^i = d_B^{i-1}s^i + s^{i+1}d_A^i$ for all i. Two homomorphisms f, g are *homotopic*, written $f \sim g$, if $f - g$ is null-homotopic. If f is null-homotopic, then so is every $h \circ f$ and $f \circ j$. Hence there is a well defined *homotopy category* $\mathbf{K}(\mathcal{A})$ whose objects are complexes, and whose arrows are homotopy classes of homomorphisms in $\mathbf{C}(\mathcal{A})$.

Remark 1.7. There is a complex $\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)$ with n-th term

$$\operatorname{Hom}^n(A^\bullet, B^\bullet) = \prod_j \operatorname{Hom}(A^j, B^{j+n}),$$

and differentials defined as follows: given an element (ϕ^j) of $\operatorname{Hom}^n(A^\bullet, B^\bullet)$, where $\phi^j\colon A^j \to B^{j+n}$ are arrows in $\mathcal{A}$, $d^n(\phi^j)$ is the element of $\operatorname{Hom}^{n+1}(A^\bullet, B^\bullet)$ whose j-th component is the arrow $(d_B^{j+n}\phi^j - (-1)^n\phi^{j+1}d_A^j)\colon A^j \to B^{j+n+1}$. An element $f \in \operatorname{Hom}^0(A^\bullet, B^\bullet)$, given by arrows $f^j\colon A^j \to B^j$, is a cycle ($f \in \ker(d^0)$) if and only if f is a homomorphism of complexes, and f is a boundary ($f \in \operatorname{im}(d^{-1})$) if and only if f is null-homotopic. Thus $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, B^\bullet) \cong H^0(\operatorname{Hom}^\bullet(A^\bullet, B^\bullet))$.

Proposition 1.8. *The following properties of a homomorphism $f\colon A \to B$ in $\mathbf{C}(\mathcal{A})$ are equivalent:*

(a) $f \sim 0$.

(b) f factors through the canonical map $i\colon A \to C(1_A)$ given by 1.5(ii) for 1_A.

(c) f factors through the canonical map $p[-1]\colon C(1_B)[-1] \to B$ given by 1.5(ii) for 1_B.

(d) The exact sequence 1.5(ii) for f splits.

Proof. Exercise. □

Corollary 1.9. *Homotopic maps $f \sim g$ induce the same maps on cohomology. In particular, the cohomology functors $\mathbf{K}(\mathcal{A}) \to \mathcal{A}$, $A \mapsto H^i(A)$ are well-defined.*

Proof. A null-homotopic map $f\colon A \to B$ induces the zero map on cohomology because it factors through $C(1_A)$, which is acyclic by 1.5(vi). □

Corollary 1.10. *Every homotopy equivalence (i.e., every homomorphism invertible in $\mathbf{K}(\mathcal{A})$) is a quasi-isomorphism.*

Remark 1.11. Proposition 1.5(vi) can be strengthened (exercise) to say that if f is bijective, then $C(f)$ is homotopy-equivalent to zero. However, it is not true that every acyclic complex is homotopy-equivalent to zero, nor do 1.5(iv, v) hold for homotopy-equivalence.

2. Triangles

The homotopy category $\mathbf{K}(\mathcal{A})$ and the derived category $\mathbf{D}(\mathcal{A})$, to be introduced in §3, are additive but not abelian categories. Instead, they share an extra structure described by a distinguished collection of *exact triangles.* Although we are mainly interested in the derived category, we first consider triangles in the homotopy category. It will be easier to deduce the main properties of the derived category after this intermediate step.

Lemma 2.1. *Given a homomorphism of complexes $f\colon A \to B$, each composite of two successive maps in the sequence*

$$\cdots \to A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \xrightarrow{f[1]} B[1] \to \cdots$$

induced by 1.5(ii) is zero in $\mathbf{K}(\mathcal{A})$.

Proof. The composite $B \to C(f) \to A[1]$ is already zero in $\mathbf{C}(\mathcal{A})$. For $A \xrightarrow{f} B \xrightarrow{i} C(f)$, the diagram

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

yields a map $C(1_A) \to C(f)$, and one checks easily that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i(1_A)}\downarrow & & \downarrow{\scriptstyle i} \\ C(1_A) & \longrightarrow & C(f) \end{array}$$

commutes. Hence $i \circ f$ is null-homotopic, by 1.8(b). A similar argument takes care of $C(f) \xrightarrow{p} A[1] \xrightarrow{f[1]} B[1]$, and the rest follows by shift-invariance. □

Definition 2.2. A *triangle* in an additive category with shift functors $A \mapsto A[n]$ is a sequence

$$A \to B \to C \to A[1]$$

for which the conclusion of 2.1 holds (with C in place of $C(f)$). A *morphism of triangles* is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & B \arrow[r] \arrow[d, "v"'] & C \arrow[r] \arrow[d, "w"'] & A[1] \arrow[d, "{u[1]}"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd} .$$

A *standard triangle* in $\mathbf{K}(\mathcal{A})$ is a triangle of the form

$$A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} A[1],$$

induced from a homomorphism $f\colon A \to B$ by 1.5(ii). An *exact triangle* in $\mathbf{K}(\mathcal{A})$ is a triangle isomorphic to a standard triangle.

Triangles are also displayed like this:

$$\begin{tikzcd} & C \arrow[dl, "+1"'] & \\ A \arrow[rr] & & B. \arrow[ul] \end{tikzcd}$$

Proposition 2.3. *Exact triangles in $\mathbf{K}(\mathcal{A})$ satisfy the following axioms:*

(o) Any triangle isomorphic to an exact triangle is exact.

(i) Every arrow $f\colon A \to B$ is the base of an exact triangle $A \xrightarrow{f} B \to C \to A[1]$. For every object A, the triangle $A \xrightarrow{1_A} A \to 0 \to A[1]$ is exact.

(ii) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is an exact triangle, then so are its left and right rotations

$$B \xrightarrow{-g} C \xrightarrow{-h} A[1] \xrightarrow{-f[1]} B[1], \quad C[-1] \xrightarrow{-h[-1]} A \xrightarrow{-f} B \xrightarrow{-g} C.$$

(iii) Given exact triangles $A \xrightarrow{f} B \to C \to A[1]$ and $A' \xrightarrow{g} B' \to C' \to A'[1]$, every commutative diagram in $\mathbf{K}(\mathcal{A})$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "h'"'] \\ A' \arrow[r, "g"] & B' \end{tikzcd}$$

extends to a morphism of triangles

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\ A' \arrow[r, "g"] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd} .$$

(iv) A direct sum of exact triangles is exact.

Proof. Axioms (o), (iv) and the first part of (i) are obvious. For the second part of (i), we have $C(1_A) \cong 0$ in $\mathbf{K}(\mathcal{A})$ by Remark 1.11.

For (ii), using shift invariance (see remark below), it suffices to verify the first rotation. We can assume the given triangle is standard, $A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} A[1]$. Then we must show that $A[1] \cong C(i)$ in $\mathbf{K}(\mathcal{A})$, via an isomorphism such that the composite $A[1] \to C(i) \xrightarrow{p(i)} B[1]$ is $-f[1]$ and $C(f) \xrightarrow{i(i)} C(i) \to A[1]$ is p. Now, $C(i)$ is identical to the mapping cone of the map $h\colon A \to C(1_B)$ obtained by composing $i(1_B)\colon B \to C(1_B)$ with f. This gives a canonical map $\pi = p(h) : C(i) \to A[1]$, by 1.5(ii). But $C(i)$ is also identical to the mapping cone of $(f, 1_B)\colon A \oplus B \to B$, whose kernel is isomorphic to A. This gives a map $\iota = k[1] : A[1] \to C(i)$, by 1.5(iii). One checks that $\pi \circ \iota = 1_{A[1]}$, $p(i) \circ \iota = -f[1]$, $\pi \circ i(i) = p$, and $\iota \circ \pi \sim 1_{C(i)}$.

For (iii), we can assume both triangles are standard. If the given diagram commutes up to a homotopy $s\colon gh \sim h'f$, you can check that $(a^{i+1}, b^i) \mapsto (h(a^{i+1}), h'(b^i) + s(a^{i+1}))$ is a homomorphism $C(f) \to C(g)$ that yields the desired morphism of triangles in $\mathbf{K}(\mathcal{A})$. □

Remarks 2.4. (a) *Warning:* The morphism of triangles in (iii) is not canonical, but depends on the choice of an isomorphism between each of the given triangles and a standard triangle. Thus we do *not* have a functorial "mapping-cone" construction assigning to each arrow $f\colon A \to B$ in $\mathbf{K}(\mathcal{A})$ an exact triangle with f as its base.

(b) Triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ and $A \xrightarrow{-f} B \xrightarrow{-g} C \xrightarrow{h} A[1]$ are isomorphic via -1_B, and likewise if we change any two signs. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is exact, a triangle such as $A \xrightarrow{-f} B \xrightarrow{-g} C \xrightarrow{-h} A[1]$ with one or three signs changed is *anti-exact*. Note that $C(f)[1] = C(-f[1])$, so the shift $A[1] \xrightarrow{f[1]} B[1] \xrightarrow{g[1]} C[1] \xrightarrow{h[1]} A[2]$ is anti-exact, while $A[1] \xrightarrow{-f[1]} B[1] \xrightarrow{-g[1]} C[1] \xrightarrow{-h[1]} A[2]$, gotten by rotating our original triangle three times, is exact.

(c) Verdier defined a *triangulated category* to be an additive category with shift functors and a class of distinguished triangles, satisfying axioms 2.3(o–iii) and an additional, more complicated, "octahedral axiom" which relates exact triangles based on f, g and $g \circ f$, and implies (iv). The fundamental examples of triangulated categories in the sense of Verdier are homotopy categories of complexes, derived categories, and the stable homotopy category of spectra of CW-complexes in topology. The logical significance of the octahedral axiom remains a bit murky. On the one hand, it is stronger than needed for the elementary applications of derived categories to supplying a good framework for homological algebra and sheaf cohomology. On the other hand, while Verdier's definition has proven adequate for the theory of triangulated categories so far, it is possible that future developments might require stronger axioms, valid in the natural examples, but not following from Verdier's axioms.

3. The derived category

Definition 3.1. The *derived category* $\mathbf{D}(\mathcal{A}) = \mathbf{C}(\mathcal{A})[Q^{-1}]$ of $\mathcal{A}$ is the category obtained from $\mathbf{C}(\mathcal{A})$ by formally inverting all quasi-isomorphisms. More precisely, $\mathbf{D}(\mathcal{A})$ is equipped with a tautological functor $j : \mathbf{C}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$, and has the universal property that given any functor $F\colon \mathbf{C}(\mathcal{A}) \to \mathcal{B}$, if F sends all quasi-isomorphisms in $\mathbf{C}(\mathcal{A})$ to isomorphisms in $\mathcal{B}$, then F factors as $F = F' \circ j$ for a unique functor $F'\colon \mathbf{D}(\mathcal{A}) \to \mathcal{B}$.

Remark 3.2. Some set-theoretic foundational issues in category theory impinge on the construction of derived categories. The objects of an ordinary category $\mathcal{C}$ need not form a set, but $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is required to be a set for all objects A, B of $\mathcal{C}$. A category whose objects do form a set is called *small*. Conversely, we may allow each $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to be a proper class, in which case $\mathcal{C}$ is called *large*.

If $\mathcal{C}$ is a small category, we can formally invert any subset Q of its arrows to get another small category $\mathcal{C}[Q^{-1}]$ with the same objects as $\mathcal{C}$, the arrows of $\mathcal{C}[Q^{-1}]$ being defined by suitable generators and relations. If $\mathcal{C}$ is an ordinary category, we can again construct $\mathcal{C}[Q^{-1}]$, but in general only as a large category.

In practice there are several strategies for coping with the set-theoretic difficulties. (1) Ignore them—as I will do in these notes. (2) Work only with small categories. Many categories of interest, such as the category of schemes of finite type over a given ring k, are equivalent to a small category. Others can be well approximated by small categories. For example, on a ringed space X, the category of sheaves of $\mathcal{O}_X$ modules of cardinality less than a fixed bound κ is equivalent to a small category. (3) Work inside a *Grothendieck universe* (an inner model of set theory). The existence of Grothendieck universes is an example of a large cardinal axiom, whose consistency with standard ZFC set theory is strictly stronger than the consistency of ZFC itself, but if you are willing to assume such axioms then this approach is a feasible solution. (4) Use Gödel-Bernays set theory, which provides explicitly for a hierarchy of proper classes beyond sets. (5) Prove that specific derived categories of interest are equivalent to ordinary categories. This is usually true and not hard to prove; for example it holds for the derived category $\mathbf{D}(\mathcal{A})$ of any abelian category which has enough injective objects. The mysterious remark in Weibel's textbook about "proving that the derived category exists in our universe" seems meant to allude to this last strategy.

In any event, these set-theoretic technicalities are beside the point: the real issues are down to earth questions like how describe an arrow in $\mathbf{D}(\mathcal{A})$, and how to recognize when two arrows are equal.

By the definition of quasi-isomorphism, the cohomology functors $H^i\colon \mathbf{C}(\mathcal{A}) \to \mathcal{A}$ factor through unique functors $H^i\colon \mathbf{D}(\mathcal{A}) \to \mathcal{A}$. If A is an object of $\mathcal{A}$, let $A[0]$ denote the complex which is A in degree zero, and 0 in other degrees. Then $H^0(A[0]) = A$, so $A \mapsto A[0]$ is a fully faithful embedding of $\mathcal{A}$ into $\mathbf{D}(\mathcal{A})$, with left inverse given by the functor H^0. Usually we just identify A with $A[0]$ and regard $\mathcal{A}$ as a full subcategory of $\mathbf{D}(\mathcal{A})$.

Proposition 3.3. *The canonical functor* $\mathbf{C}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ *factors (uniquely) through* $\mathbf{K}(\mathcal{A})$.

Proof. It suffices to prove that $f \sim 0$ implies $f = 0$ in $\mathbf{D}(\mathcal{A})$. This is immediate from 1.8(b), since $C(1_A) \cong 0$ in $\mathbf{D}(\mathcal{A})$, by 1.5(vi). $\square$

Corollary 3.4. *The derived category* $\mathbf{D}(\mathcal{A})$ *can also be identified with* $\mathbf{K}(\mathcal{A})[Q^{-1}]$.

Remarks 3.5. (a) Traditionally, $\mathbf{D}(\mathcal{A})$ is often defined as $\mathbf{K}(\mathcal{A})[Q^{-1}]$. This tends to overemphasize the role of the homotopy category, which is not essential to the definition, although it is a useful auxiliary device for understanding many properties of $\mathbf{D}(\mathcal{A})$.

(b) Equality in $\mathbf{D}(\mathcal{A})$ of homomorphisms $f, g \in \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(A, B)$ does *not* imply that f and g are homotopic. A criterion for equality of arrows in the derived category is given by 3.22(ii), below.

Definition 3.6. An *exact triangle* in $\mathbf{D}(\mathcal{A})$ is a triangle isomorphic in $\mathbf{D}(\mathcal{A})$ to a standard triangle, as in 2.2. Equivalently (by 1.10), a triangle in $\mathbf{D}(\mathcal{A})$ is exact iff it is isomorphic in $\mathbf{D}(\mathcal{A})$ to an exact triangle of $\mathbf{K}(\mathcal{A})$.

An advantage of the derived category is that every exact sequence in $\mathbf{C}(\mathcal{A})$ gives rise to an exact triangle in $\mathbf{D}(\mathcal{A})$, which is not the case in $\mathbf{K}(\mathcal{A})$.

Proposition 3.7. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence in* $\mathbf{C}(\mathcal{A})$. *Then the diagram*

$$\begin{array}{ccc} C(f) & \underset{qis}{\overset{\simeq}{\to}}{}^{q(f)} & C \\ {\scriptstyle p(f)}\downarrow & & \downarrow{\scriptstyle i(g)} \\ A[1] & \underset{qis}{\overset{\simeq}{\to}}{}^{k(g)[1]} & C(g) \end{array}$$

anti-commutes in $\mathbf{K}(\mathcal{A})$, *and hence in* $\mathbf{D}(\mathcal{A})$.

Proof. The formula for $k(g)[1] \circ p(f)$ is $(a^{i+1}, b^i) \mapsto (f(a^{i+1}), 0)$, and for $i(g) \circ q(f)$ it is $(a^{i+1}, b^i) \mapsto (0, g(b^i))$. Then $s^i(a^{i+1}, b^i) = (b^i, 0)$ is a homotopy between $k(g)[1] \circ p(f)$ and $-i(g) \circ q(f)$. $\square$

Definition 3.8. The map $h\colon C \to A[1]$ in $\mathbf{D}(\mathcal{A})$, given in terms of the diagram in 3.7 by $h = p(f) \circ q(f)^{-1} = -k(g)[1]^{-1} \circ i(g)$, is the *connecting homomorphism* of the exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$.

Proposition 3.9. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence in* $\mathbf{C}(\mathcal{A})$. *There is an exact triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1],$$

in $\mathbf{D}(\mathcal{A})$, *where* h *is the connecting homomorphism.*

Proof. By 1.5(iv), $q(f)\colon C(f) \to C$ is a quasi-isomorphism whose composite with the canonical map $i\colon B \to C(f)$ is g. Hence the standard triangle based on f is isomorphic in $\mathbf{D}(\mathcal{A})$ to the triangle above. $\square$

Remark 3.10. By similar reasoning, the standard triangle based on the map g in the exact sequence in 3.9 is isomorphic to

$$B \xrightarrow{g} C \xrightarrow{-h} A[1] \xrightarrow{f[1]} B[1],$$

or to the same triangle with f, g changed to $-f$, $-g$. Rotating this triangle gives the one in 3.9. The fact that the diagram in 3.7 anticommutes, rather than commutes, is precisely what is needed to ensure that the two ways of making an exact sequence in $\mathbf{C}(\mathcal{A})$ into a standard triangle in $\mathbf{D}(\mathcal{A})$ are consistent with each other.

Lemma 3.11. *Exact triangles in $\mathbf{D}(\mathcal{A})$ satisfy the rotation axiom 2.3(ii).*

This is obvious from the definition and 2.3(ii) for $\mathbf{K}(\mathcal{A})$. Below we will see that in fact all the axioms 2.3(o–iv) hold in $\mathbf{D}(\mathcal{A})$. First we need two preliminaries: the cohomology long exact sequence, which is a basic tool for all of homological algebra, and a description of the arrows in $\mathbf{D}(\mathcal{A})$.

Proposition 3.12. *If $A \to B \to C \to A[1]$ is an exact triangle in $\mathbf{D}(\mathcal{A})$—in particular, if $0 \to A \to B \to C \to 0$ is an exact sequence of complexes and $h\colon C \to A[1]$ is the connecting homomorphism in $\mathbf{D}(\mathcal{A})$—there is an induced long exact sequence of cohomology groups*

$$\cdots \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to \cdots .$$

Proof. Apply 3.11 and 1.5(vii). □

Corollary 3.13. *If a morphism between exact triangles in $\mathbf{D}(\mathcal{A})$ is an isomorphism at two of the three corners of the triangle, then it is an isomorphism of triangles.*

Proof. An arrow in $\mathbf{D}(\mathcal{A})$ is an isomorphism iff it induces isomorphisms in cohomology. In the diagram of long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^i(A) & \longrightarrow & H^i(B) & \longrightarrow & H^i(C) & \longrightarrow & H^{i+1}(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^i(A') & \longrightarrow & H^i(B') & \longrightarrow & H^i(C) & \longrightarrow & H^{i+1}(A') & \longrightarrow & \cdots \end{array},$$

every third vertical arrow is an isomorphism, given that the others are. □

Corollary 3.14. *Let $A \xrightarrow{f} B \to C \to A[1]$ be an exact triangle (in either $\mathbf{D}(\mathcal{A})$ or $\mathbf{K}(\mathcal{A})$) based on a homomorphism $f\colon A \to B$ in $\mathbf{C}(\mathcal{A})$. Then f is a quasi-isomorphism if and only if C is acyclic.*

Proposition 3.15. *If $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is an exact triangle in $\mathbf{K}(\mathcal{A})$, then for any complex X, there are induced long exact sequences*

$$\cdots \to \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, A) \to \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, B) \to \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, C) \to \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, A[1]) \to \cdots$$
$$\cdots \leftarrow \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A, X) \leftarrow \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(B, X) \leftarrow \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(C, X) \leftarrow \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A[1], X) \leftarrow \cdots$$

Proof. We prove the second sequence; a similar argument applies to the first. Suppose $f\colon B \to X$ satisfies $fu \sim 0$. From 2.3(i,iii) we get a morphism of triangles

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & B \arrow[r, "v"] \arrow[d, "f"'] & C \arrow[r, "w"] \arrow[d, "g"'] & A[1] \arrow[d] \\ 0 \arrow[r] & X \arrow[r, "1_X"] & X \arrow[r] & 0 \end{tikzcd},$$

which shows $f \sim gv$ for some $g\colon C \to X$. In other words, $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, X) \leftarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(B, X) \leftarrow \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(C, X)$ is exact, and the rest follows by the rotation axiom, 2.3(ii). $\square$

Note that in this proof we only used the fact that $\mathbf{K}(\mathcal{A})$ satisfies the axioms 2.3. Hence 3.15 holds in any 'weak' triangulated category (as defined by these axioms).

Corollary 3.16. *Corollary 3.13 also holds in* $\mathbf{K}(\mathcal{A})$ *and every weak triangulated category.*

Proof. In the proof of 3.13 we can use either long exact sequence in 3.15 in place of the one in 3.12, together with the fact that the functor $\mathrm{Hom}(-, C)$ (resp. $\mathrm{Hom}(C, -)$) determines C up to canonical isomorphism. $\square$

Our next goal is to describe the arrows in $\mathbf{D}(\mathcal{A})$.

Definition 3.17. A class of Q of arrows in a category $\mathcal{C}$ is a (left) *Ore system* if it satisfies the following conditions:

(a) Q is multiplicative, *i.e.* $Q \circ Q \subseteq Q$ and $1_X \in Q$ for every object X of $\mathcal{C}$.

(b) Every pair of arrows $A' \xleftarrow{q} A \xrightarrow{f} B$ with $q \in Q$ can be completed to a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "r"'] \\ A' \arrow[r, "g"] & B' \end{tikzcd}$$

with $r \in Q$.

(c) If $f \circ q = 0$ with $q \in Q$, there exists $r \in Q$ such that $r \circ f = 0$.

A *right* Ore system is defined dually.

Remark 3.18. A category is *filtered* if every pair $f : A \to B$, $f' : A \to B'$ of morphisms from the same object A can be completed to a commutative square. An inductive system of sets $(X_i)_{i \in I}$ indexed by a filtered category I is also said to be *filtered*. A filtered inductive system has the property that elements $x \in X_i$, $x' \in X_{i'}$ represent the same element of the direct limit $\varinjlim(X_i)$ if and only if there exist arrows $\alpha\colon X_i \to X_j$, $\alpha'\colon X_{i'} \to X_j$ in I such that $\alpha(x) = \alpha'(x')$. To see this, one checks that in a filtered inductive system, the preceding condition defines an equivalence relation $x \equiv x'$, and then $\varinjlim(X_i) = (\bigsqcup_{i \in I} X_i)/\equiv$.

Let Q be a left Ore system in $\mathcal{C}$, fix an object A in $\mathcal{C}$, and let $Q\backslash A$ be the category of arrows $A \xrightarrow{q} A'$, where $q \in Q$, with morphisms the commutative triangles

$$\begin{array}{ccc} A & \xrightarrow{q} & A' \\ & \searrow_{q'} & \downarrow \\ & & A'' \end{array} .$$

Then (a–c) imply (exercise) that $Q\backslash A$ is a filtered category.

Proposition 3.19. *Assume Q is a left Ore system in $\mathcal{C}$. Then morphisms in $\mathcal{C}[Q^{-1}]$ are given by the filtered direct limits*

$$\mathrm{Hom}_{\mathcal{C}[Q^{-1}]}(A, B) = \varinjlim_{B \xrightarrow{q} B'} \mathrm{Hom}_{\mathcal{C}}(A, B').$$

Denoting an element $A \xrightarrow{f} B' \xleftarrow{q} B$ of this direct limit by $q^{-1}f$, the composition law is given by $(s^{-1}f) \circ (q^{-1}h) = (rs)^{-1}(gh)$, where $B' \xleftarrow{q} B \xrightarrow{f} C'$ completes as in (b) to a diagram such that $gq = rf$.

Proof. Using 3.17 (c), one verifies that any two diagram completions $A' \xrightarrow{g} B' \xleftarrow{r} B$, $A' \xrightarrow{g'} B'' \xleftarrow{r'} B$ in (b) represent the same element $r^{-1}g = r'^{-1}g'$ of $\lim_{B \to B'} \mathrm{Hom}_{\mathcal{C}}(A', B')$. This given, we can define a category $\mathcal{C}'$ with $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \lim_{B \to B'} \mathrm{Hom}_{\mathcal{C}}(A, B')$, and check that the composition law specified in the proposition is well-defined and associative. There is an obvious functor $j\colon \mathcal{C} \to \mathcal{C}'$ sending $f\colon A \to B$ to $1_B^{-1} f$, and is immediate that for any $q\colon A \to B$ in Q, $j(q)$ has inverse $q^{-1}1_B$. It is also immediate that $\mathcal{C}'$ has the universal property of $\mathcal{C}[Q^{-1}]$. Namely, given a functor $F\colon \mathcal{C} \to \mathcal{B}$ such that $F(q)$ is invertible for all $q \in Q$, F extends to $\mathcal{C}'$ by $F(q^{-1}f) = F(q)^{-1}F(f)$, which is easily seen to be independent of the choice of representative $q^{-1}f$. □

Lemma 3.20. *The quasi-isomorphisms form a left and right Ore system in $\mathbf{K}(\mathcal{A})$.*

Proof. By duality, "left" suffices. Condition (a) is trivial. For (b) take B' to be the mapping cone of $(q, f)\colon A \to A' \oplus B$, with $(g, r)\colon A' \oplus B \to B'$ the canonical map i in 1.5(ii). This mapping cone is identical to the mapping cone of the map $h\colon C(q)[-1] \to B$ given by composing f with the canonical map $p[-1]\colon C(q)[-1] \to A$. The map $r\colon B \to B'$ coincides under this identification with the canonical map $B \to C(h)$. By 3.14, $C(q)$ is acyclic, hence r is a quasi-isomorphism by another application of 3.14.

For (c), given $A' \xrightarrow{q} A \xrightarrow{f} B$, let $C = C(q)$, $i\colon A \to C$ the canonical map. Since $fq = 0$ in $\mathbf{K}(\mathcal{A})$, the second long exact sequence in 3.15 implies that $f = gi$ for some $g\colon C \to B$. Let $B' = C(g)$, $r\colon B \to B'$ the canonical map $i(g)$. Now, C is acyclic by 3.14, hence r is a quasi-isomorphism by 3.14 again. Finally, $rg = 0$ in $\mathbf{K}(\mathcal{A})$ by 2.1, hence $rf = rgi = 0$. □

Remark 3.21. The quasi-isomorphisms satisfy conditions (a) and (b) for an Ore system in $\mathbf{C}(\mathcal{A})$, but we need to work in $\mathbf{K}(\mathcal{A})$ to have (c) hold.

Corollary 3.22. *(i) Every arrow in* $\mathbf{D}(\mathcal{A})$ *factors as* $q^{-1}f$ *and as* gr^{-1}*, where* f*,* g*,* q*,* r *are homomorphisms in* $\mathbf{C}(\mathcal{A})$*, with* q*,* r *quasi-isomorphisms.*

(ii) A homomorphism $f\colon A \to B$ *in* $\mathbf{C}(\mathcal{A})$ *is zero in* $\mathbf{D}(\mathcal{A})$ *if and only if the equivalent conditions hold: (a) there exists a quasi-isomorphism* $q\colon B \to B'$ *such that* $qf \sim 0$*; (b) there exists a quasi-isomorphism* $r\colon A' \to A$ *such that* $fr \sim 0$*.*

Corollary 3.23. *The exact triangles in* $\mathbf{D}(\mathcal{A})$ *satisfy axioms 2.3(o–iv).*

Proof. Axiom (o) holds by definition, (iv) is clear, and we have already seen (ii). Axioms (i) and (iii) follow easily from the corresponding axioms in $\mathbf{K}(\mathcal{A})$, using 3.22(i). □

Corollary 3.24. *Proposition 3.15 also holds in* $\mathbf{D}(\mathcal{A})$*.*

Proof. As noted following the proof of 3.15, the proposition holds in every weak triangulated category. □

Corollary 3.25. *The exact triangle based on an arrow* $f\colon A \to B$ *in* $\mathbf{D}(\mathcal{A})$ *is unique up to (non-canonical) isomorphism.*

Proof. Follows from axiom (iii) and 3.13. □

Remarks 3.26. (a) The octahedral axiom follows similarly, so $\mathbf{D}(\mathcal{A})$ is a triangulated category in the sense of Verdier.

(b) The reasoning employed above applies more generally. Let $\mathcal{K}$ be a triangulated category and $\mathcal{N} \subseteq \mathcal{K}$ a full triangulated subcategory, closed under isomorphisms in $\mathcal{K}$. Let Q consist of the arrows in $\mathcal{K}$ such that the third vertex of any exact triangle based on $q \in Q$ is an object of $\mathcal{N}$. Then Q is a left and right Ore system in $\mathcal{K}$, and $\mathcal{D} = \mathcal{K}[Q^{-1}]$ is a triangulated category, also denoted $\mathcal{D} = \mathcal{K}/\mathcal{N}$. In our case, $\mathcal{K} = \mathbf{K}(\mathcal{A})$, with $\mathcal{N}$ consisting of the acyclic complexes. By 3.14, this is equivalent to Q consisting of the quasi-isomorphisms.

4. Derived Functors

We will use Deligne's method [5] of defining and constructing derived functors.

Definition 4.1. Given a complex A, let $qis\backslash A$ be the category of quasi-isomorphisms $A \underset{qis}{\xrightarrow{\simeq}} A'$ in $\mathbf{K}(\mathcal{A})$, with morphisms the commutative triangles as in 3.18.

By 3.18, $qis\backslash A$ is a filtered category. Given a functor $F\colon \mathbf{K}(\mathcal{A}) \to \mathcal{C}$ and an object Y of $\mathcal{C}$, we have a filtered inductive limit of sets, functorial in Y,

$$\varinjlim_{A \underset{qis}{\xrightarrow{\simeq}} A'} \big(\mathrm{Hom}_{\mathcal{C}}(Y, F(A'))\big).$$

Proposition 4.2. *Let* $F\colon \mathbf{K}(\mathcal{A}) \to \mathcal{C}$ *be any functor. To each arrow* $f\colon A \to B$ *in* $\mathbf{K}(\mathcal{A})$ *there is canonically associated a natural transformation*

$$r_F(f)\colon \varinjlim_{A \underset{qis}{\xrightarrow{\simeq}} A'} \big(\mathrm{Hom}_{\mathcal{C}}(-, F(A'))\big) \to \varinjlim_{B \underset{qis}{\xrightarrow{\simeq}} B'} \big(\mathrm{Hom}_{\mathcal{C}}(-, F(B'))\big) \tag{1}$$

between functors from $\mathcal{C}^{\mathrm{op}}$ *to* $\underline{\mathrm{Sets}}$*. This gives a functor* $r_F\colon \mathbf{K}(\mathcal{A}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Sets}})$*.*

Proof. Recall (3.20) that the quasi-isomorphisms $Q \subseteq \mathbf{K}(\mathcal{A})$ form a left Ore system (3.17). Given $A \xrightarrow{q} A'$ in $qis\backslash A$, we can define a an arrow $F(A') \to F(B')$ in $\mathcal{C}$, for some B' in $qis\backslash B$, by completing the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow \\ A' & \xrightarrow{g} & B' \end{array} \tag{2}$$

in $\mathbf{K}(\mathcal{A})$ and applying F to the bottom row. Equivalently, this defines a natural map $\rho_{A'}\colon \mathrm{Hom}_{\mathcal{C}}(-, F(A')) \to \mathrm{Hom}_{\mathcal{C}}(-, F(B'))$. As in the proof of 3.19, any two completed diagrams (2) factor into a third. Hence the natural map $\mathrm{Hom}_{\mathcal{C}}(-, F(A')) \to \lim_{B \underset{qis}{\xrightarrow{\simeq}} B'} \big(\mathrm{Hom}_{\mathcal{C}}(-, F(B'))\big)$ represented by $\rho_{A'}$ is independent of the choice of completion (2). Since $\rho_{A'}$ is functorial with respect to A' in $qis\backslash A$, these maps combine to give (1). One checks easily that $r_F(f)$ defined this way is functorial in f. □

Definition 4.3. Any category $\mathcal{C}$ has a fully faithful *Yoneda embedding* $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Sets}})$ given by $X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$. The functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Sets}})$ has direct limits, inherited from $\underline{\mathrm{Sets}}$. The closure under filtered direct limits of the image of $\mathcal{C}$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Sets}})$ is called the *ind-completion* $\mathrm{Ind}(\mathcal{C})$ of $\mathcal{C}$.

The dual concept, constructed from the dual Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \underline{\mathrm{Sets}})^{\mathrm{op}}$, $X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, -)$, is the *pro-completion* $\mathrm{Pro}(\mathcal{C})$ of $\mathcal{C}$. (In other words, $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}} = \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$.)

Remark 4.4. (a) $\mathrm{Ind}(\mathcal{C})$ has the universal property that any functor $F\colon \mathcal{C} \to \mathcal{D}$ into a category $\mathcal{D}$ with filtered direct limits extends uniquely to a functor $\widetilde{F}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ which preserves filtered direct limits. In particular, any functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor $\mathrm{Ind}(F)\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ commuting with the inclusions, and if $\mathcal{C}$ has filtered direct limits, the inclusion has a canonical left inverse $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$.

(b) The condition for a filtered inductive system (A_λ) in $\mathcal{C}$ to have $\varinjlim_\lambda (A_\lambda) = X$ in $\mathrm{Ind}(\mathcal{C})$, for an object X in $\mathcal{C}$, is that there is a functorial isomorphism $\mathrm{Hom}(-, X) = \varinjlim_\lambda \mathrm{Hom}(-, A_\lambda)$.

Note that this is *not* the same thing as (A_λ) having an inductive limit $\varinjlim_\lambda (A_\lambda) = X$ in $\mathcal{C}$, which by definition means that there is a functorial isomorphism $\mathrm{Hom}(X, -) = \varprojlim_\lambda \mathrm{Hom}(A_\lambda, -)$.

The condition $\mathrm{Hom}(-, X) = \varinjlim_\lambda \mathrm{Hom}(-, A_\lambda)$ in fact is stronger than $\varinjlim_\lambda (A_\lambda) = X$. It is equivalent to the existence of a system of maps $p_\lambda\colon A_\lambda \to X$, compatible with all arrows in (A_λ), satisfying the following conditions: (i) for some indices λ cofinal in the given inductive system, there are morphisms $j_\lambda\colon X \to A_\lambda$ such that $p_\lambda j_\lambda = 1_X$ and the j_λ are compatible with arrows in (A_λ); (ii) for every such λ, there is an arrow $r\colon A_\lambda \to A_\mu$ that equalizes 1_{A_λ} and $j_\lambda p_\lambda$, that is, $r = r j_\lambda p_\lambda$. Under these conditions one can verify that the p_λ make X an inductive limit $\varinjlim_\lambda (A_\lambda) = X$, but in general an inductive limit need not arise in this way.

In fact, every functor automatically preserves conditions (i) and (ii), but not every functor preserves inductive limits.

In the above language, (1) defines a functor $r_F\colon \mathbf{K}(\mathcal{A}) \to \operatorname{Ind}(\mathcal{C})$. When F is the identity functor, we obtain in particular a canonical functor $j = r_{1_{\mathbf{K}(\mathcal{A})}}\colon \mathbf{K}(\mathcal{A}) \to \operatorname{Ind}(\mathbf{K}(\mathcal{A}))$ sending A to the ind-object "$\varinjlim$"$(qis\backslash A) = \lim_{\substack{A \xrightarrow{\cong} A' \\ qis}} (\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(-, A'))$. For arbitrary F, we have $r_F = \operatorname{Ind}(F) \circ j$.

Proposition 4.5. *The image of the functor $j : \mathbf{K}(\mathcal{A}) \to \operatorname{Ind}(\mathbf{K}(\mathcal{A}))$, $A \mapsto$ "$\varinjlim$"$(qis\backslash A)$ is isomorphic to $\mathbf{D}(\mathcal{A})$, with j corresponding to the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.*

Proof. The definitions having been understood, this is merely a restatement of 3.19. □

Now let $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be an *exact functor* of triangulated categories, *i.e.* an additive functor which commutes with shifts and preserves exact triangles. In particular, any additive functor $F\colon \mathcal{A} \to \mathcal{B}$ induces an exact functor $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ (also denoted F by abuse of notation). In practice, we only deal with functors of this last form.

Composing the canonical functor $j_{\mathcal{B}}\colon \mathbf{K}(\mathcal{B}) \to \mathbf{D}(\mathcal{B})$ with F, and applying 4.2–4.5 (with $\mathcal{C} = \mathbf{D}(\mathcal{B})$), we define a functor $RF = r_{j_{\mathcal{B}}F}\colon \mathbf{D}(\mathcal{A}) \to \operatorname{Ind}(\mathbf{D}(\mathcal{B}))$.

Definition 4.6. The *right derived functor of F is defined at A* in $\mathbf{D}(\mathcal{A})$, with *value X* in $\mathbf{D}(\mathcal{B})$, if $RF(A) = X$ belongs to $\mathbf{D}(\mathcal{B}) \subseteq \operatorname{Ind}(\mathbf{D}(\mathcal{B}))$.

It is an exercise for the reader to work out the dual definition of *left derived functor LF*.

The practical meaning of 4.6 will become clearer in §5, where we will give criteria that one uses in practice to show that $RF(A)$ is defined. The criteria also have the effect of making $RF(A)$ concretely computable, often in more than one way. But first we need to remain a little longer in the abstract context in order to establish the basic properties of RF.

Definition 4.7. The cohomology objects $H^i(RF)$ are denoted R^iF and called the *classical right derived functors of F.*

Corollary 4.8. *Let $\mathbf{D}^F(\mathcal{A})$ be the full subcategory of $\mathbf{D}(\mathcal{A})$ whose objects are those A such that $RF(A)$ is defined. Then RF is a functor from $\mathbf{D}^F(\mathcal{A})$ to $\mathbf{D}(\mathcal{B})$.*

Corollary 4.9. *The subcategory $\mathbf{D}^F(\mathcal{A})$ is closed under isomorphisms in $\mathbf{D}(\mathcal{A})$.*

Corollary 4.10. *If $RF(A)$ is defined then $RF(A[n])$ is defined and equal to $RF(A)[n]$.*

Corollary 4.11. *Suppose F maps quasi-isomorphisms to quasi-isomorphisms (in particular, if F comes from an exact functor $F\colon \mathcal{A} \to \mathcal{B}$). Then RF is defined on all of $\mathbf{D}(\mathcal{A})$ and is just the functor from $\mathbf{D}(\mathcal{A})$ to $\mathbf{D}(\mathcal{B})$ induced by F, via the universal property of $\mathbf{D}(\mathcal{A})$.*

Proof. In this case, $F(qis\backslash A)$ is a constant inductive system in $\mathbf{D}(\mathcal{B})$ with limit $F(A)$. □

Remark 4.12. Originally, Verdier defined a right derived functor of F (assuming one exists) to be a functor $RF\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$, together with a natural transformation $j_{\mathcal{B}} \circ F \to RF \circ j_{\mathcal{A}}$, satisfying the universal property that for any other such functor $G\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$, the transformation $j_{\mathcal{B}} \circ F \to G \circ j_{\mathcal{A}}$ factors through $\theta \circ j_{\mathcal{A}}$ for a unique natural transformation $\theta\colon RF \to G$. Verdier's definition is still the one found most often in the literature.

Historically, it was not always clear how to construct some important derived functors on all of $\mathbf{D}(\mathcal{A})$, so Verdier also allowed derived functors on a subcategory $\mathcal{D} \subseteq \mathbf{D}(\mathcal{A})$ (for instance, on $\mathbf{D}^+(\mathcal{A})$—see 5.4), defined by the same universal property, restricted to functors from $\mathcal{D}$ to $\mathbf{D}(\mathcal{B})$.

By construction, Deligne's RF has Verdier's universal property, but now among functors $\mathbf{D}(\mathcal{A}) \to \mathrm{Ind}(\mathbf{D}(\mathcal{B}))$. When Deligne's RF is defined on $\mathcal{D} \subseteq \mathbf{D}(\mathcal{A})$, it is then immediate that it is a right derived functor of F in the sense of Verdier.

It is not clear that existence of a Verdier derived functor, say $\mathcal{R}F$, must imply that RF is defined everywhere (whence $RF = \mathcal{R}F$). As it turns out, this question is unimportant, because in practice, the techniques one uses to construct a Verdier derived functor on $\mathcal{D}$ actually show that Deligne's RF is defined on $\mathcal{D}$.

Theorem 4.13 (Deligne [5])**.** *Let $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be exact (e.g., if F comes from an additive functor $F\colon \mathcal{A} \to \mathcal{B}$). Let $A \to B \to C \to A[1]$ be an exact triangle in $\mathbf{D}(\mathcal{A})$. If $RF(A)$ and $RF(B)$ are defined, then $RF(C)$ is defined, and $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ is an exact triangle in $\mathbf{D}(\mathcal{B})$.*

We need the following lemma for the proof.

Lemma 4.14. *Let $A \to B \to C \to A[1]$ be an exact triangle in $\mathbf{K}(\mathcal{A})$. There exist morphisms of exact triangles in $\mathbf{K}(\mathcal{A})$*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle q[1]}\downarrow \\ A' & \xrightarrow{u} & B' & \xrightarrow{v} & C' & \xrightarrow{w} & A'[1] \end{array} \tag{3}$$

with all vertical arrows quasi-isomorphisms, such that A', B', C' are cofinal in $qis\backslash A$, $qis\backslash B$, $qis\backslash C$, respectively.

Proof. Given (3) and $B' \xrightarrow{r'} B''$ in $qis\backslash B$, there is an exact triangle $A' \to B'' \to C'' \to A'[1]$ on $r'u\colon A' \to B''$, and, by 2.3(iii), a morphism from $A' \to B' \to C' \to A'[1]$ to $A' \to B'' \to C'' \to A'[1]$, whose component arrows are all quasi-isomorphisms, by 3.13. This shows that the vertices B' in (3) are cofinal in $qis\backslash B$. The corresponding statement holds for the other vertices by 2.3(ii). $\square$

Proof of Theorem 4.13. Let $RF(A) = X$, $RF(B) = Y$, and complete the arrow $X \to Y$ to an exact triangle $X \to Y \to Z \to X[1]$ in $\mathbf{D}(\mathcal{B})$.

By Remark 4.4(b), if $RF(A) = X$, there exist quasi-isomorphisms $A \to A'$, cofinal in $qis\backslash A$, such that the canonical map $F(A') \to X = RF(A)$ has a section $j_{A'}\colon X \to F(A')$. Applying this to $Y = RF(B)$ as well, we can find sections $j_{A'}\colon X \to F(A')$, $j_{B'}\colon Y \to F(B')$.

Changing A', B' as needed, we can fit these into a commutative diagram in $\mathbf{D}(\mathcal{B})$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle j_{A'}}\big\downarrow & & {\scriptstyle j_{B'}}\big\downarrow \\ F(A') & \xrightarrow{F(u)} & F(B') \end{array},$$

and we can further assume that $u\colon A' \to B'$ is part of an exact triangle in the bottom row of (3). Since F is exact, this extends to a morphism of exact triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ F(A') & \longrightarrow & F(B') & \longrightarrow & F(C') & \longrightarrow & F(A')[1] \end{array}. \tag{4}$$

Now, 4.14 and 3.24 imply that the sequence

$$\begin{aligned} \cdots \to \varinjlim_{A'} \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(T, F(A')) &\to \varinjlim_{B'} \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(T, F(B')) \\ &\to \varinjlim_{C'} \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(T, F(C')) \to \varinjlim_{A'} \operatorname{Hom}_{\mathbf{D}(\mathcal{B})}(T, F(A')[1]) \to \cdots \end{aligned} \tag{5}$$

is exact for every object T of $\mathbf{D}(\mathcal{B})$. By definition, this is just the sequence

$$\begin{array}{ccccccccc} \cdots \to & RF(A) & \to & RF(B) & \to & RF(C) & \to & RF(A)[1] & \to \cdots \\ & || & & || & & & & || & \\ & \operatorname{Hom}(-, X) & & \operatorname{Hom}(-, Y) & & & & \operatorname{Hom}(-X[1]) & \end{array}$$

in $\operatorname{Ind}(\mathbf{D}(\mathcal{B}))$, evaluated at T. The morphism in (4) provides a commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & \operatorname{Hom}(-, X) & \to & \operatorname{Hom}(-, Y) & \to & \operatorname{Hom}(-, Z) & \to & \operatorname{Hom}(-, X[1]) & \to \cdots \\ & || & & || & & \downarrow & & || & \\ \cdots \to & RF(A) & \to & RF(B) & \to & RF(C) & \to & RF(A)[1] & \to \cdots \end{array}.$$

Evaluated at any T in $\mathbf{D}(\mathcal{B})$, we have just seen that the bottom row is exact, and the top row is exact by 3.24. Hence the vertical arrow is an isomorphism. $\square$

Corollary 4.15. *For any triangle $A \to B \to C \to A[1]$ in $\mathbf{D}(\mathcal{A})$, and in particular, for any exact sequence $0 \to A \to B \to C \to 0$ in $\mathbf{C}(\mathcal{A})$, if $RF(A)$, $RF(B)$, $RF(C)$ are defined (e.g., if any two of them are), there is a long exact sequence of classical derived functors*

$$\cdots \to R^0F(A) \to R^0F(B) \to R^0F(C) \to R^1F(A) \to R^1F(B) \to \cdots.$$

5. Computing derived functors

Definition 5.1. Let $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be an exact functor. For any complex I in $\mathbf{K}(\mathcal{A})$, there is a canonical morphism $\eta\colon F(I) \to RF(I)$ in $\operatorname{Ind}(\mathbf{D}(\mathcal{B}))$, that is, a functorial map from the functor represented by $F(I)$ to the functor that defines $RF(I)$. If η is an isomorphism, then of course $RF(I)$ is defined and given by $F(I)$. In this case we say that I is *expanded* for F.[1]

A quasi-isomorphism $A \xrightarrow{q} I$ is called a *resolution of* A. If I is expanded for F, then $RF(A)$ is defined and can be identified with $F(I)$, the natural morphism $F(A) \to RF(A)$ being given by $F(q)$. We will give some criteria for the existence of a resolution of A that is expanded for F.

Proposition 5.2. *If F sends all quasi-isomorphisms $A \to A'$ to quasi-isomorphisms, then A is expanded for F.*

Proof. In this case, $F(qis\backslash A)$ is a constant inductive system (all its maps are isomorphisms in $\mathbf{D}(\mathcal{B})$), with limit $F(A)$. $\square$

In particular, this gives another way to see 4.11.

Lemma 5.3. *Let $\mathfrak{I} \subseteq qis\backslash A$ be a class of complexes such that*

(i) For every resolution $A \underset{qis}{\xrightarrow{\simeq}} A'$ there is a resolution $A' \underset{qis}{\xrightarrow{\simeq}} I$ with $I \in \mathfrak{I}$, i.e., $\mathfrak{I}$ is cofinal in $qis\backslash A$, and in particular, A has a resolution $A \underset{qis}{\xrightarrow{\simeq}} I$ with $I \in \mathfrak{I}$;

(ii) For every quasi-isomorphism $q\colon I \underset{qis}{\xrightarrow{\simeq}} J$ with $I, J \in \mathfrak{I}$, $F(q)$ is a quasi-isomorphism.

Then every $I \in \mathfrak{I}$ is expanded for F. In particular, a resolution $A \underset{qis}{\xrightarrow{\simeq}} I$ induces $F(A) \xrightarrow{F(q)} RF(A) = F(I)$.

Proof. By hypothesis $F(\mathfrak{I})$ is cofinal in $F(qis\backslash A)$ and constant with limit $F(I)$. $\square$

Definition 5.4. The *bounded-below* derived category $\mathbf{D}^+(\mathcal{A})$ is the full subcategory of $\mathbf{D}(\mathcal{A})$ consisting of objects A such that for some n_0, we have $H^i(A) = 0$ for all $i < n_0$. The *bounded-above* derived category $\mathbf{D}^-(\mathcal{A})$ is defined dually. The *bounded* derived category is $\mathbf{D}^b(\mathcal{A}) = \mathbf{D}^+(\mathcal{A}) \cap \mathbf{D}^-(\mathcal{A})$.

Remark 5.5. The *truncation functor* $\tau_{\geq n}$ sends a complex A to the complex

$$\tau_{\geq n}(A) = \cdots \to 0 \to (A^n/\operatorname{im}(d^{n-1})) \to A^{n+1} \to A^{n+2} \to \cdots .$$

Then

$$H^i(\tau_{\geq n}(A)) = \begin{cases} H^i(A) & \text{if } i \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\tau_{\geq n}$ preserves quasi-isomorphisms and hence is well-defined as an endo-functor on any of the categories $\mathbf{C}(\mathcal{A})$, $\mathbf{K}(\mathcal{A})$, $\mathbf{D}(\mathcal{A})$. There is an obvious canonical functorial surjection $A \to \tau_{\geq n}(A)$. The exact sequence $0 \to \tau_{<n}(A) \to A \to \tau_{\geq n}(A) \to 0$ defines the

[1] Spaltenstein [8] translates Deligne's phrase '*déployé pour F*' as 'unfolded for F,' but 'expanded' seems more accurate in this context.

dual truncation functor $\tau_{<n}$ which kills the cohomology $H^{\geq n}(A)$. In $\mathbf{D}(\mathcal{A})$, this becomes a triangle

$$\tau_{<n}(A) \to A \to \tau_{\geq n}(A) \to \tau_{<n}(A)[1] = \tau_{<(n-1)}(A[1]).$$

If A is already bounded below at n_0, then $A \to \tau_{\geq n_0}(A)$ is a quasi-isomorphism. It follows immediately that $\mathbf{D}^+(\mathcal{A})$ is equivalent to its full subcategory of *strictly bounded-below* objects A, satisfying $A^i = 0$ for all i less than some n_0. Moreover, if $A \to X \underset{qis}{\overset{\simeq}{\leftarrow}} B$ is a morphism between objects $A, B \in \mathbf{D}^+(\mathcal{A})$, then necessarily $X \in \mathbf{D}^+(\mathcal{A})$. Truncating all three objects, we can replace X by a strictly bounded-below complex too. Hence $\mathbf{D}^+(\mathcal{A})$ can be identified with the localization $\mathbf{C}^+(\mathcal{A})[Q^{-1}]$ of the category $\mathbf{C}^+(\mathcal{A})$ of strictly bounded-below complexes by the quasi-isomorphisms in $\mathbf{C}^+(\mathcal{A})$.

Proposition 5.6. *Assume $F\colon \mathcal{A} \to \mathcal{B}$ is left exact. Let $\mathfrak{A}$ be a class of objects in $\mathcal{A}$ such that*

(i) For every A in $\mathcal{A}$ there is an injection $A \to I$ with $I \in \mathfrak{A}$;

(ii) $\mathfrak{A}$ is closed under finite direct sums, and if $0 \to I \to J \to N \to 0$ is an exact sequence with $I, J \in \mathfrak{A}$, then $N \in \mathfrak{A}$;

(iii) If $0 \to I \to A \to B \to 0$ is an exact sequence with $I \in \mathfrak{A}$, then $0 \to F(I) \to F(A) \to F(B) \to 0$ is exact.

Then every A in $\mathbf{D}^+(\mathcal{A})$ has a resolution $A \to I^\bullet$, where $I^\bullet$ is a strictly bounded-below complex of objects in $\mathfrak{A}$, and any such $I^\bullet$ is expanded for F, i.e., the resolution induces $F(A) \to RF(A) = F(I^\bullet)$.

Proof. We can assume that A is strictly bounded-below, say $A^i = 0$ for $i < 0$. Suppose by induction on k that we can construct a homomorphism of complexes

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A^k \arrow[r] \arrow[d] & A^{k+1} \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^k \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd},$$

where the I^j belong to $\mathfrak{A}$, and the induced maps $H^i(A) \to H^i(I)$ are isomorphisms for $i < k$ and injective for $i = k$ (initially, we have this with $k = -1$).

Let M be the cokernel of $A^k \to A^{k+1} \oplus I^k/\partial(I^{k-1})$. Then in the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A^k \arrow[r] \arrow[d] & A^{k+1} \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^k \arrow[r] & M \arrow[r] & \cdots \end{tikzcd}, \tag{6}$$

the bottom row is a complex and the vertical arrows give a homomorphism of complexes. The co-fiber square

$$\begin{tikzcd} A^k \arrow[r] \arrow[d] & A^{k+1} \arrow[d] \\ I^k/\partial(I^{k-1}) \arrow[r] & M \end{tikzcd}.$$

induces an isomorphism between the cokernels of the rows, which implies that (6) induces an injection on H^i for $i = k+1$. From the co-fiber square we also get a surjection from the kernel of the top row to the kernel of the bottom row, which implies that that (6) induces a surjection on H^k. But we had an injection on H^k before adjoining M to the bottom row, and we still have this because $\partial(I^{k-1})$ is unchanged. Hence in (6) we have an isomorphism on H^k.

Finally, we can replace M in (6) with an object $I^{k+1} \in \mathfrak{A}$ for which we have an injection $M \to I^{k+1}$. This preserves the condition that the induced maps on H^i are isomorphisms for $i \leq k$ and injective for $i = k+1$. Hence, by induction, we can extend the bottom row for all k, obtaining in the end a resolution $A \underset{qis}{\xrightarrow{\simeq}} I$ by a bounded below complex of objects in $\mathfrak{A}$.

Let $I \xrightarrow{q} J$ be a quasi-isomorphism between bounded-below complexes of objects in $\mathfrak{A}$. Then $C(q)$ is a bounded-below acyclic complex of objects in $\mathfrak{A}$, and from (ii, iii) it follows easily by induction on the cohomology degree that $F(C(q)) = C(F(q))$ is acyclic. Hence $F(q)$ is a quasi-isomorphism. By the first part of the proof, every resolution of I has a strictly bounded-below resolution J by a complex of objects in $\mathfrak{A}$. Then Lemma 5.3 shows that I is expanded for F. □

Objects A in a class $\mathfrak{A}$ satisfying 5.6(i–iii) are said to be *acyclic* for F. It follows from Proposition 5.6 that they satisfy $R^iF(A) = 0$ for all $i > 0$. Conversely, using Corollary 4.15, one sees that the class $\mathfrak{A}$ of *all* objects A such that $R^iF(A) = 0$ for all $i > 0$ satisfies (i–iii). The value of Proposition 5.6 is that it enables us to recognize a class of acyclic objects without knowing how to calculate RF in advance.

Example 5.7. Let $\mathcal{A}$ be the category of (left) R-modules for any (possibly non-commutative) ring R, or the category of sheaves of $\mathcal{O}_X$-modules, where X is a ringed space. Then every object A of $\mathcal{A}$ has an injection $A \to I$ into an injective object. Any exact sequence as in 5.6(iii) with I injective splits, and injective objects satisfy 5.6(ii). Hence the injectives satisfy (i–iii) for any left-exact functor F. We conclude that RF is always defined on $\mathbf{D}^+(\mathcal{A})$, and that $RF(A) = F(I)$, where I is a strictly bounded-below injective resolution of A.

Remark 5.8. Spaltenstein [8] defines a complex I in $\mathbf{C}(\mathcal{A})$ to be *K-injective* if it satisfies the following equivalent conditions:

(i) for every acyclic complex A in $\mathbf{C}(\mathcal{A})$, $\mathrm{Hom}^\bullet(A, I)$ is acyclic;

(ii) if A is any acyclic complex, then every homomorphism $f : A \to I$ is null-homotopic.

(iii) every quasi-isomorphism $I \to J$ has a left homotopy inverse.

If I consists of a single object, condition (i) says that $\mathrm{Hom}(-, I)$ is an exact functor, that is, I is an injective object.

Spaltenstein shows that an inverse limit of K-injective complexes is K-injective if the inverse system satisfies a suitable Mittag-Leffler condition. This implies that every bounded-below complex of injectives in $\mathcal{A}$ is K-injective; hence, as above, if $\mathcal{A}$ has enough injectives, then every bounded-below complex has a K-injective resolution. Better still, if $\mathcal{A}$ has enough injectives and suitable inverse limits, Spaltenstein shows that every complex (unbounded, in general) has a K-injective resolution.

Condition (iii) and Lemma 5.3 imply that a K-injective complex is expanded for every functor. Hence, for suitable categories $\mathcal{A}$, including R-modules and sheaves of $\mathcal{O}_X$-modules, RF is defined on all of $\mathbf{D}(\mathcal{A})$ for every F, with $RF(A) = F(I)$ for any K-injective resolution $A \to I$. In this situation we also see that $\mathbf{D}(\mathcal{A})$ is equivalent to an ordinary category, namely the full subcategory of $\mathbf{K}(\mathcal{A})$ whose objects are the K-injective complexes.

Example 5.9. Let $f\colon X \to Y$ be a morphism of ringed spaces. A sheaf $\mathcal{F}$ on X is called *flasque* if the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for every open $U \subseteq X$. It is not hard to show that flasque sheaves satisfy 5.6(ii, iii) for the direct image functor f_*. For any sheaf $\mathcal{F}$, the germ maps define an injective sheaf homomorphism $\mathcal{F} \hookrightarrow \widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}}(U) = \prod_{x\in U} \mathcal{F}_x$, with restriction ρ^U_V in $\widehat{\mathcal{F}}$ defined by projection on the factors for $x \in V$. The sheaf $\widehat{\mathcal{F}}$ is clearly flasque, so flasque sheaves satify (i) (one can also verify (i) by proving that injective sheaves are flasque). Then 5.6 implies that flasque sheaves are acyclic for f_*, and that for any bounded-below complex of sheaves A on X, we have $Rf_*(A) = f_*(J)$, where J is a strictly bounded-below flasque resolution.

Proposition 5.10. *Given morphisms of ringed spaces $f\colon X \to Y$, $g\colon Y \to Z$, there is a natural isomorphism of functors*

$$Rg_* \circ Rf_* \cong R(gf)_* \tag{7}$$

from $\mathbf{D}^+(\mathcal{A})$ to $\mathbf{D}^+(\mathcal{C})$, where $\mathcal{A}$ (resp. $\mathcal{C}$) is the cateogory of sheaves of $\mathcal{O}_X$ modules (resp. $\mathcal{O}_Z$ modules).

Proof. It is immediate from the definition that any direct image $f_*\mathcal{F}$ of a flasque sheaf $\mathcal{F}$ is flasque (a property not shared by injective sheaves). Given a complex A of sheaves on X, the instance of (7) at A follows by taking a flasque resolution $A \to J$ and observing that, since f_*J is then a flasque resolution of Rf_*A, the complex $(gf)_*J = g_*f_*J$ represents both $R(gf)_*A$ and Rg_*Rf_*A. □

If g is the tautological morphism from Y to the space Z with one point and $\mathcal{O}_Z$ the constant sheaf $\mathbb{Z}$, then the functors g_* and Γ_Y are essentially the same, and similarly for $(gf)_*$ and Γ_X. The following corollary is therefore a special case of 5.10.

Corollary 5.11. *Given a morphism of ringed spaces $f\colon X \to Y$, there is a natural isomorphism of functors*

$$R\Gamma_Y \circ Rf_* \cong R\Gamma_X \tag{8}$$

from the derived category $\mathbf{D}^+(\mathcal{A})$, where $\mathcal{A}$ is the category of sheaves of $\mathcal{O}_X$ modules, to the derived category of sheaves of abelian groups (or, more generally, of R modules, if f is a morphism of ringed spaces over the one-point ringed space Z with $\mathcal{O}_Z(Z) = R$).

The analog of (7) for classical derived functors is a spectral sequence relating the functors $R^pf_* \circ R^qg_*$ and $R^n(gf)_*$. The identity $Rg_* \circ Rf_* \cong R(gf)_*$ is not only simpler than the old-fashioned spectral sequence, it is a stronger result. Spaltenstein [8] gave a definition of *K-flasque complex* of sheaves. Using this, he proved that 5.10 and 5.11 also hold for unbounded complexes.

Proposition 5.12. *Let $f\colon X \to Y$ be a morphism of ringed spaces. Let $\mathcal{B}$ be a base of the topology on Y in the weak sense, that is, $\mathcal{B}$ is a set of open subsets $V \subseteq Y$ such that every open $U \subseteq Y$ is a union of members of $\mathcal{B}$, but we do not require that $\mathcal{B}$ be closed under finite intersections.*

For a sheaf M of $\mathcal{O}_X$ modules to be acyclic for the direct image functor f_*, it suffices that M be acylic for $R\Gamma_{f^{-1}(V)}$ for all $V \in \mathcal{B}$.

Proof. Let $\mathfrak{A}$ be the class of sheaves A on X satisfying the acyclicity condition we require of M. Explicitly, A belongs to $\mathfrak{A}$ if for all $V \in \mathcal{B}$, the canonical map $\Gamma(f^{-1}(V), A) \to R\Gamma(f^{-1}(V), A)$ is a quasi-isomorphism, or equivalently, $R^i\Gamma(f^{-1}(V), A) = 0$ for all $i > 0$. We verify that $\mathfrak{A}$ satisfies the conditions in 5.6 for the functor $F = f_*$. Condition (i) follows because flasque sheaves belong to $\mathfrak{A}$. Condition (ii) follows from the long exact sequence 3.12 for $R\Gamma_{f^{-1}(V)}$. For condition (iii), suppose $0 \to I \to A \to B \to 0$ is exact and $I \in \mathfrak{A}$. Then 3.12 implies that $(f_*A)(V) \to (f_*B)(V)$ is surjective for all $V \in \mathcal{B}$. Since $\mathcal{B}$ contains a (weak) base of open neighborhoods of every point $y \in Y$, this implies that $f_*A \to f_*B$ is surjective. $\square$

As a final application of 5.6, let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{A}$ the category of sheaves of $\mathcal{O}_X$ modules, and $\mathcal{B}$ the category of sheaves of abelian groups on X. If M is a sheaf of $\mathcal{O}_X$ modules, then $\Gamma_X(M)$ is a module for the ring $R = \mathcal{O}_X(X)$, but we can forget this structure if we wish and view Γ_X as a functor from $\mathcal{A}$ to abelian groups. On this view, Γ_X is the composite of the global section functor on $\mathcal{B}$ with the with the forgetful functor $j\colon \mathcal{A} \to \mathcal{B}$. An important basic fact is that either way of viewing Γ_X leads to the same derived functor.

Proposition 5.13. *Let j denote the forgetful functor from sheaves of $\mathcal{O}_X$ modules to sheaves of abelian groups. Since it is an exact functor, we also denote its derived functor by j. Let Γ_X denote the global section functor on sheaves of abelian groups on X, and $\Gamma' = \Gamma_X \circ j$ the global section functor from sheaves of $\mathcal{O}_X$ modules to abelian groups. Then $R(\Gamma') = (R\Gamma_X) \circ j$ as a functor from $\mathbf{D}^+(\mathcal{O}_X\text{-Mod})$ to the derived category of abelian groups.*

Proof. Let A be a complex of sheaves of $\mathcal{O}_X$ modules. Replacing A by a flasque resolution I, we have $\Gamma'(I) = \Gamma_X(j(I))$ and $j(I)$ is a flasque resolution of $j(A)$. $\square$