## Synopsis of material from EGA Chapter IV, §§4.1-4.6

### 4. Base change for algebraic preschemes

**4.1. Dimension of algebraic preschemes.**

Dimension theory for general preschemes is in §5. A more elementary version is given here in the algebraic case.

The notation $\operatorname{deg.tr}_K(L)$ refers to the transcendence degree of a field extension.

*Definition* (4.1.1). — Let $X$ be a prescheme locally of finite type over a field $k$. We define the dimension of $X$ to be

$$\dim X = \sup_x \operatorname{deg.tr}_k \mathbf{k}(x) \tag{4.1.1.1}$$

for $x$ among the maximal points of $X$.

We will see later (5.2.2) that $\dim X$ only seems to depend on the base field $k$, and coincides with the topological dimension of the underlying space of $X$. We clearly have $\dim X = \dim X_{\mathrm{red}}$.

Since each $\mathbf{k}(x)$ is a finitely generated extension of $k$, $\operatorname{deg.tr}_k \mathbf{k}(x)$ is finite. If $X$ is of finite type, then, being Noetherian, it has a finite number of irreducible components, so $\dim X$ is finite. For the empty variety, we set

$$\dim(\emptyset) = -\infty.$$

If $(X_\alpha)$ is the family of induced reduced subschemes on the irreducible components of $X$ (I, 5.2.1), then

$$\dim(X) = \sup_\alpha \dim(X_\alpha). \tag{4.1.1.2}$$

This reduces the computation of the dimension to the case of integral preschemes locally of finite type over $k$.

We also have

$$\dim(X) = \dim(U) \tag{4.1.1.3}$$

for any dense open $U \subseteq X$. This ultimately reduces the notion of dimension to the case of affine schemes of finite type over $k$.

*Theorem* (4.1.2). — *Let $f\colon X \to Y$ be a $k$-morphism of preschemes locally of finite type over a field $k$.*

*(i) If $f$ is quasi-compact and dominant, then* $\dim(Y) \leq \dim(X)$.

*(ii) If $f$ is quasi-finite, then* $\dim(X) \leq \dim(Y)$.

*(iii) Suppose $X$ is of finite type over $k$. A necessary and sufficient condition to have* $\dim(X) \geq n$ *(resp.* $\leq n$, *resp.* $= n$*) is that there exist a dense open $U \subseteq X$ and a surjective (resp. finite, resp. finite and surjective) $k$-morphism $g\colon U \to \mathbb{A}^n_k$. [Nowadays* $\mathbb{A}^n_k = \operatorname{Spec}(k[T_1, \ldots, T_n])$ *is standard notation, but EGA uses $\mathbf{V}(k^n)$ or $\mathbf{V}^n_k$ here instead.]*

*Corollary* (4.1.2.1). — *Let $Y$ be a $k$-prescheme locally of finite type. For every subprescheme $Z \subseteq Y$ we have* $\dim(Z) \leq \dim(Y)$*. If all irreducible components of $Y$ have the same dimension [$Y$ is* equidimensional*], then* $\dim(Z) < \dim(Y)$ *if and only if the complement of $Z$ is dense in $Y$.*

[The proof of (i) is easy; (ii) reduces to Corollary (4.1.2.1); the proof of the latter and (iii) are based on the Noether normalization lemma.]

*Remark* (4.1.3). — Corollary (4.1.2.1) implies that formula (4.1.1.1) also holds with $x$ ranging over *all* points of $X$.

*Corollary* (4.1.4). — *Let $X$ be a prescheme locally of finite type over $k$, and $k \subseteq K$ a field extension. Then* $\dim(X \otimes_k K) = \dim(X)$*.*

*Corollary* (4.1.5). — *Let $X$ and $Y$ be a preschemes locally of finite type over a field $k$. Then* $\dim(X \times_k Y) = \dim(X) + \dim(Y)$*.*

## 4.2. **Associated prime cycles on algebraic preschemes.**

*Proposition* (4.2.1). — *Let $K$ and $L$ be extensions of a field $k$, such that $K \otimes_k L$ is Noetherian. Then the associated prime ideals of $K \otimes_k L$ are all minimal, and if $E$ is the residue field at such an ideal, we have*

$$\deg.\mathrm{tr}_K E = \deg.\mathrm{tr}_k L, \qquad \deg.\mathrm{tr}_L E = \deg.\mathrm{tr}_k K, \tag{4.2.1.1}$$

*and hence*

$$\deg.\mathrm{tr}_k E = \deg.\mathrm{tr}_k K + \deg.\mathrm{tr}_k L. \tag{4.2.1.2}$$

*Corollary* (4.2.2). — *Under the hypotheses of (3.3.6), if the preschemes $T_{x,y}$ are locally Noetherian, they have no embedded associated prime cycles.*

*Corollary* (4.2.3). — *Under the hypotheses of (3.3.6) (resp. (3.3.7)), if the $T_{x,y}$ are locally Noetherian, and if $\mathcal{F}$ and the $\mathcal{G}_s$ for $s \in S$ (resp. $\mathcal{F}$ and $\mathcal{G}$) have no embedded associated prime cycles, then neither does $\mathcal{F} \otimes_S \mathcal{G}$.*

*Proposition* (4.2.4). — *Let $k$ be a field and $X, Y$ locally Noetherian $k$-preschemes such that $X \otimes_k Y$ is locally Noetherian. Suppose further that $X$ and $Y$ are integral. Then:*

*(i) $X \times_k Y$ has no embedded associated prime cycles, each irreducible component of $X \times_k Y$ dominates $X$ and $Y$, and these components are in bijective correspondence with those of* $\mathrm{Spec}(R(X) \otimes_k R(Y))$ *(that is, with the minimal primes of $R(X) \otimes_k R(Y)$), where $R(X), R(Y)$ are the fields of rational functions on $X, Y$.*

*(ii) Given a maximal point $z \in X \times_k Y$ corresponding to a minimal prime $\mathfrak{p}$ of $R(X) \otimes_k R(Y)$, the local ring $\mathcal{O}_{X \times_k Y}, z$ is isomorphic to the localization $(R(X) \otimes_k R(Y))_{\mathfrak{p}}$. In particular, if either $R(X)$ or $R(Y)$ is separable over $k$, then $X \times_k Y$ is reduced.*

*(iii) If, in addition, $X$ and $Y$ are locally of finite type over $k$, then every irreducible component of $X \times_k Y$ has dimension* $\dim(X) + \dim(Y)$*.*

*Proposition* (4.2.5). — *Let $k$ be a field, $X, Y$ locally Noetherian $k$-preschemes, $\mathcal{F}$ (resp. $\mathcal{G}$) a quasi-coherent $\mathcal{O}_X$-Module (resp. $\mathcal{O}_Y$-Module). Let $(Z'_\lambda)$ (resp. $(Z''_\mu)$) be the family of associated prime cycles of $\mathcal{F}$ (resp. $\mathcal{G}$), or, using the same notation, the induced reduced subschemes on these cycles. Then, if $Z'_\lambda \times_k Z''_\mu$ is locally Noetherian, the irreducible components $Z_{\lambda\mu\nu}$ of $Z'_\lambda \times_k Z''_\mu$ dominate $Z'_\lambda$ and $Z''_\mu$, and $(Z_{\lambda\mu\nu})$ is the family of distinct associated prime cycles of $\mathcal{F} \otimes_k \mathcal{G}$.*

*Corollary* (4.2.6). — *Let $k$ be a field and $X, Y$ locally Noetherian $k$-preschemes such that $X \times_k Y$ is locally Noetherian. Let $(Z'_\lambda)$ (resp. $(Z''_\mu)$) be the family of induced reduced subschemes on the irreducible components of $X$ (resp. $Y$). Then the irreducible components $Z_{\lambda\mu\nu}$ of $Z'_\lambda \times_k Z''_\mu$ dominate $Z'_\lambda$, and $Z''_\mu$ and $(Z_{\lambda\mu\nu})$ is the family of irreducible components of $X \times_k Y$.*

*Proposition* (4.2.7). — *Let $k \subseteq K$ be a field extension and $X$ a $k$-prescheme such that $X \otimes_k K$ is locally Noetherian, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-Module, $x'$ a point of $X \otimes_k K$, and $x$ its image in $X$.*

*(i) Let $(Z_\lambda)$ be the family of induced reduced subschemes on the associated prime cycles of $\mathcal{F}$. Then the irreducible components $Z_{\lambda\mu}$ of the $Z_\lambda \otimes_k K$ are the associated prime cycles of $\mathcal{F} \otimes_k K$, and $Z_{\lambda\mu}$ dominates $Z_\lambda$; moreover $Z_{\lambda\mu}$ is embedded if and only if $Z_\lambda$ is.*

*(ii) The point $x$ belongs to an embedded associated prime cycle of $\mathcal{F}$ if and only if $x'$ belongs to an embedded associated prime cycle of $\mathcal{F} \otimes_k K$; and $\mathcal{F}$ has no embedded associated prime cycles if and only if the same is true for $\mathcal{F} \otimes_k K$.*

*(iii) The $\lambda$ such that $x \in Z_\lambda$ are those for which there exists an index $\mu$ such that $x' \in Z_{\lambda\mu}$. In particular, if $x'$ belongs to a unique associated prime cycle of $\mathcal{F} \otimes_k K$, then $x$ belongs to a unique associated prime cycle of $\mathcal{F}$.*

*(iv) If $X$ is locally of finite type over $k$, then $\dim(Z_{\lambda\mu}) = \dim(Z_\lambda)$.*

*Corollary* (4.2.8). — *If $X$ is locally of finite type over $k$, the set of dimensions of associated prime cycles is the same for $\mathcal{F}$ and $\mathcal{F} \otimes_k K$. The set of dimensions of irreducible components is the same for $X$ and $X \otimes_k K$.*

*Proposition* (4.2.9). — *Given the hypotheses of (4.2.5), suppose further that $\mathcal{F}$ and $\mathcal{G}$ are coherent. Let $(\mathcal{F}_\lambda)_{\lambda \in L}$ and $(\mathcal{G}_\mu)_{\mu \in M}$ be irredundant decompositions of $\mathcal{F}$ and $\mathcal{G}$. For each $(\lambda, \mu) \in L \times M$, let $(\mathcal{K}_{\lambda\mu\nu})_{\nu \in S(\lambda,\mu)}$ be a reduced irredundant decomposition of $\mathcal{F}_\lambda \otimes_k \mathcal{G}_\mu$, where $S(\lambda, \mu) = \operatorname{Ass}(\mathcal{F}_\lambda \otimes_k \mathcal{G}_\mu)$ (3.2.5). Then $(\mathcal{K}_{\lambda\mu\nu})$, over all the triples $(\lambda, \mu, \nu)$, is an irredundant decomposition of $\mathcal{F} \otimes_k \mathcal{G}$, reduced if $(\mathcal{F}_\lambda)$ and $(\mathcal{G}_\mu)$ are.*

*Corollary* (4.2.10). — *Under the hypotheses of (4.2.7), suppose further that $\mathcal{F}$ is coherent, and let $(\mathcal{F}_\lambda)_{\lambda \in L}$ be an irredundant decompostion of $\mathcal{F}$. For each $\lambda \in L$, let $(\mathcal{F}_{\lambda\mu})_{\mu \in \operatorname{Ass}(\mathcal{F}_\lambda \otimes_k K)}$ be an irredundant decomposition of $\mathcal{F}_\lambda \otimes_k K$. Then $(\mathcal{F}_{\lambda\mu})$ is an irredundant decomposition of $\mathcal{F} \otimes_k K$, reduced if $(\mathcal{F}_\lambda)$ is.*

### 4.3. **Review on tensor products of fields.**

(4.3.1). A field extension $k \subseteq L$ is *primary* if the largest separable algebraic extension of $k$ in $L$ is $k$ itself.

*Proposition* (4.3.2). — *Let $k \subseteq K, L$ be field extensions. If $k \subseteq L$ is primary, then* $\operatorname{Spec}(L \otimes_k K)$ *is irreducible, and if $\xi$ is its generic point, then $\mathbf{k}(\xi)$ is a primary extension of $K$. Conversely, if* $\operatorname{Spec}(L \otimes_k K)$ *is irreducible for every $k \subseteq K$, then $k \subseteq L$ is a primary extension.*

*Corollary* (4.3.3). — *If $k$ is separably closed (*i.e.*, if its algebraic closure is radicial over $k$), then* $\operatorname{Spec}(L \otimes_k K)$ *is irreducible for all extensions $k \subseteq K, L$, and conversely.*

*Corollary* (4.3.4). — *Let $k \subseteq L$ be a field extension and $L_s$ the separable algebraic closure of $k$ in $L$, that is, the largest separable algebraic extension of $k$ contained in $L$. Suppose the degree $[L_s : k]$ is finite, and let $k \subseteq K$ be a Galois extension containing $L_s$. Then* $\operatorname{Spec}(L \otimes_k K)$ *has $[L_s : k]$ irreducible components, they are disjoint, and the residue field at each of their generic points is a primary extension of $K$.*

*Proposition* (4.3.5). — *Let $k \subseteq K, L$ be field extensions. If $k \subseteq L$ is separable, then* $\operatorname{Spec}(L \otimes_k K)$ *is reduced. Conversely, if* $\operatorname{Spec}(L \otimes_k K)$ *is reduced for every radicial extension $k \subseteq K$, then $k \subseteq L$ is a separable extension.*

*Corollary* (4.3.6). — *If $k$ is perfect, then* $\operatorname{Spec}(L \otimes_k K)$ *is reduced for all extensions $k \subseteq K, L$, and conversely.*

*Corollary* (4.3.7). — *Let $k \subseteq K, L$ be field extensions such that $k \subseteq L$ is separable and either $K$ or $L$ is finite over $k$. Then the residue fields of the semi-local ring $L \otimes_k K$ are separable extensions of $K$.*

*Corollary* (4.3.8). — *If $k \subseteq K, L$ are finite separable field extensions, then the ring $L \otimes_k K$ is a direct product of finitely many separable extensions of $k$.*

*Proposition* (4.3.9). — *If $k$ is algebraically closed, then $L \otimes_k K$ is an integral domain for all extensions $k \subseteq K, L$, and conversely.*

## 4.4. **Irreducible and connected preschemes over an algebraically closed field.**

(4.4.1). Let $k \subseteq K$ be a field extension and $X$ a $k$-prescheme. Using (2.4.9), (2.2.13) and (2.3.5), one shows that each irreducible component of $X \otimes_k K$ dominates an irreducible component of $X$, and the the resulting map between the sets of irreducible components is surjective. Since $p\colon X \otimes_k K \to X$ is surjective and continuous, it also induces a surjective map between the sets of connected components.

Each irreducible (resp. connected) component $Z'$ of $X \otimes_k K$ is therefore contained in a unique set $p^{-1}(Z)$, where $Z$ is an irreducible (resp. connected) component of $X$. Denoting also by $Z$ any subscheme of $X$ with underlying space $Z$, we have that $Z'$ is an irreducible (resp. connected) component of $Z \otimes_k K$.

If $X \otimes_k K$ has a finite number $n$ (resp. $n'$) of irreducible (resp. connected) components—for instance, if $X$ is of finite type over $k$, in which case $X \otimes_k K$ is of finite type over $K$ and thus Noetherian—then $X$ has at most $n$ (resp. $n'$) irreducible (resp. connected) components, with equality holding if and only if, for the induced reduced prescheme $Z$ on each irreducible (resp. connected) component of $X$, we have $Z \otimes_k K$ irreducible (resp. connected).

In particular if $X \otimes_k K$ is irreducible (resp. connected), then so is $X$. If $X \otimes_k K$ is reduced (resp. integral), then so is $X$ because $p$ is faithfully flat (2.1.13).

In this section and the next we study more closely how the irreducible or connected components of $X \otimes_k K$ vary with $K$; what we have just seen is that the number of components increases with extension of fields.

*Lemma* (4.4.2). — *Let $f\colon X' \to X$ be a continuous map of topological spaces satisfying the conditions:*

*(i) $f$ is open and surjective (resp. $f$ is such that $X$ can be identified with the quotient space of $X'$ by the equivalence relation defined by $f$).*

*(ii) For all $x \in X$, $f^{-1}(X)$ is irreducible (resp. connected).*

*Then $X'$ is irreducible (resp. connected) if and only if $X$ is.*

*Remark* (4.4.3). — If $X$ does not have the quotient topology, $X$ and every $f^{-1}(x)$ can even be irreducible without $X'$ being connected. Example: $X = \mathbb{A}^1_k$, $X'$ is the disjoint union of $x$ and $X - x$ for a closed point $x \in X$.

If $X$ has the quotient topology, but $f$ is not open, $X$ and every $f^{-1}(x)$ can be irreducible without $X'$ being so. Example: $X' = (\mathbb{A}^1_k \times \{y\}) \cup (\{x\} \times \mathbb{P}^1_k) \subseteq \mathbb{A}^1_k \times_k \mathbb{P}^1_k$, and $f$ is the projection on $X = \mathbb{A}^1_k$. Then $f$ is surjective and proper, hence closed, so $X$ has the quotient topology, every fiber is either a point or $\mathbb{P}^1_k$, but $X'$ is reducible.

*Theorem* (4.4.4). — *Let $k$ be an algebraically closed field and $X$ a $k$-prescheme. If $X$ is irreducible (resp. connected), then so is $X \otimes_k K$ for every extension $k \subseteq K$.*

*Corollary* (4.4.5). — *Let $k$ be an algebraically closed field, $X$ a $k$-prescheme, $K$ an extension of $k$, and $p\colon X \otimes_k K \to X$ the canonical projection. If $Z$ is an irreducible (resp. connected) subset of $X$, then so is $p^{-1}(Z)$. In particular, if $X_0$ is an irreducible (resp. connected) component of $X$ containing $Z$, then $p^{-1}(X_0)$ is an irreducible (resp. connected) component of $X \otimes_k K$ containing $p^{-1}(Z)$.*

*Corollary* (4.4.6). — *With the hypotheses and notation of (4.4.5), the map $Z \mapsto p^{-1}(Z)$ is a bijection from the set of irreducible (resp. connected) components of $X$ to that of $X'$, with inverse $Z' \mapsto p(Z')$.*

## 4.5. **Geometrically irreducible and geometrically connected preschemes.**

*Proposition* (4.5.1). — *Let $k$ be a field, $X$ a $k$-prescheme, $\Omega$ an algebraically closed extension of $k$. The cardinality $n$ (resp. $n'$) of the set of irreducible (resp. connected) components of $X \otimes_k \Omega$ is independent of $\Omega$. For any extension $K$ of $k$, the cardinality of the set of irreducible (resp. connected) components of $X \otimes_k K$ is $\leq n$ (resp. $\leq n'$).*

*Definition* (4.5.2). — The cardinal $n$ (resp. $n'$) in (4.5.1) is called the *geometric number of irreducible components* (resp. *connected components*) of $X$, relative to $k$. If $n = 1$ (resp. $n' = 1$), we say that $X$ is a *geometrically irreducible* (resp. *geometrically connected*) $k$-prescheme.

By (4.5.1), the following are equivalent:

(a) $X$ is geometrically irreducible (resp. geometrically connected).
(b) For some algebraically closed extension $k \subseteq \Omega$, $X \otimes_k \Omega$ is irreducible (resp. connected).
(c) For every extension $k \subseteq K$, $X \otimes_k K$ is irreducible (resp. connected).

(4.5.3). Let $X$ be a $k$-prescheme and $Z \subseteq X$ a locally closed subset, or any subscheme with underlying space $Z$. By (I, 5.1.8), for any extension $k \subseteq K$, the number of irreducible (resp. connected) components of $Z \otimes_k K$ depends only on the subspace $Z$. Thus we can define the geometric number of irreducible or connected components of $Z$ to be that of any sub-prescheme of $X$ having underlying space $Z$.

*Proposition* (4.5.4). — *Let $f\colon X \to Y$ be a surjective $k$-morphism of $k$-preschemes. If $X$ is geometrically irreducible (resp. geometrically connected), then so is $Y$.*

*Definition* (4.5.5). — A morphism of preschemes $f\colon X \to Y$ is *irreducible* (resp. *connected*) if the $\mathbf{k}(y)$-prescheme $f^{-1}(y)$ is geometrically irreducible (resp. connected) for every $y \in Y$.
[For clarity most people now prefer to say that *the fibers of $f$ are geometrically irreducible* (resp. *geometrically connected*).]

*Proposition* (4.5.6). — *(i) Let $X$ be a $k$-prescheme and $k \subseteq K$ a field extension. Then the geometric number of irreducible (resp. connected) components of $X_{(K)}$ relative to $K$ is equal to that of $X$ relative to $k$. In particular, the $K$-prescheme $X_{(K)}$ is geometrically irreducible (resp. connected) if and only if the $k$-prescheme $X$ is.*

*Proposition* (4.5.7). — *Let $f\colon X \to Y$ be a surjective morphism with geometrically irreducible (resp. connected) fibers, let $Y' \to Y$ be any morphism, and set $X' = X \times_Y Y'$. If $Y'$ is also irreducible (resp. connected) and $f$ is universally open (resp. flat and quasi-compact, or universally open, or universally closed), then $X'$ is irreducible (resp. connected).*

*Corollary* (4.5.8). — *Let $X$ and $Y$ be $k$-preschemes.*
*(i) If $X$ is geometrically irreducible (resp. geometrically connected) and $Y$ is irreducible (resp. connected), then $X \times_k Y$ is irreducible (resp. connected).*
*(ii) If $X$ and $Y$ are geometrically irreducible (resp. geometrically connected), then so is $X \times_k Y$.*

*Proposition* (4.5.9). — *The following conditions on a $k$-prescheme $X$ are equivalent:*
*(a) $X$ is geometrically irreducible (that is, $X \otimes_k K$ is irreducible for every extension $k \subseteq K$).*
*(b) $X \otimes_k K$ is irreducible for every finite separable extension $k \subseteq K$.*
*(c) $X$ is irreducible, and $\mathbf{k}(x)$ is a primary extension of $k$, where $x$ is the generic point of $X$.*

*Corollary* (4.5.10). — *Let $X$ be an irreducible $k$-prescheme with generic point $x$, $k'$ the separable algebraic closure of $k$ in $\mathbf{k}(x)$, and $k''$ a Galois extension of $k$ (not necessarily of finite degree) containing $k'$. Suppose that $[k' : k]$ is finite. Then the irreducible components of $X \otimes_k k''$ are geometrically irreducible, the number of them is equal to $[k' : k]$, and this is also the geometric number of irreducible components of $X$.*

*Corollary* (4.5.11). — *Let $X$ be a $k$-prescheme. Suppose that $X$ has only a finite number of maximal points $x_i$ $(1 \leq i \leq r)$, and that for each $i$, the separable algebraic closure $k_i'$ of $k$ in $\mathbf{k}(x_i)$ is of finite degree $[k_i' : k]$. Then there exists a finite separable extension $k \subseteq L$ such that the irreducible components of $X \otimes_k L$ are geometrically irreducible; their number is equal to $\sum_i [k_i' : k]$ and is the geometric number of irreducible components of $X$.*

In particular, the hypotheses in (4.5.11) are satisfied when $X$ is of finite type over $k$.

*Remarks* (4.5.12). — (i) The notions in (4.5.2) depend on the base field $k$. When $k$ needs to be specified, one may use abbreviated terminology such as "$k$-irreducible" instead of "geometrically irreducible relative to $k$." The reader is cautioned that such abbreviations are used in a sense opposite to the older use of the same terms by Weil. From Weil's point of view, a variety $X'$ is given over an algebraic closed field $\Omega$ in the first place. If $X'$ happens to be identified with $X \otimes_k \Omega$ for a subfield $k$ of $\Omega$, Weil would use the term "$k$-irreducible" to mean that $X$ is irreducible.

(ii) There is no analogue of (4.5.9) for geometric connectedness. Example: let $X = \operatorname{Spec}(\mathbb{R}[[T, U]]/(T^2 + U^2))$ and let $X' \subseteq X$ be the open subscheme complementary to the closed point of $X$. Then $X$ and $X'$ are integral schemes, with the same field $\mathbf{k}(x)$ at the generic point, but $X \otimes_{\mathbb{R}} \mathbb{C}$ is connected, with two irreducible components intersecting only at the closed point, while $X' \otimes_{\mathbb{R}} \mathbb{C}$ is disconnected.

*Proposition* (4.5.13). — *Let $f \colon Y \to X$ be a $k$-morphism of $k$-preschemes. Suppose that $Y$ is non-empty and geometrically connected, and $X$ is connected. Then $X$ is geometrically connected.*

*Corollary* (4.5.13.1). — *(i) Let $f \colon Y \to X$ be a $k$-morphism of $k$-preschemes. If $Y$ is non-empty and geometrically connected, then the connected component $X_0$ of $X$ containing $f(Y)$ is geometrically connected.*

*(ii) Let $X$ be a $k$-prescheme. If $Y$ is an irreducible component of $X$ which is geometrically irreducible, then the connected component $X_0$ of $X$ containing $Y$ is geometrically connected.*

*Corollary* (4.5.14). — *Let $X$ be a $k$-prescheme. If $x$ is a point of $X$ such that $\mathbf{k}(x)$ is a primary extension of $k$ (in particular, if $x$ is a $k$-rational point of $X$), then the connected component of $X$ containing $x$ is geometrically connected.*

*Proposition* (4.5.15). — *Let $X$ be a $k$-prescheme, $x$ a point of $X$, $k'$ the separable algebraic closure of $k$ in $\mathbf{k}(x)$. Suppose that*

*(i) $X$ is connected, and*

*(ii) $k'$ is a finite extension of $k$.*

*Then the geometric number of connected components of $X$ is $\leq [k' : k]$, and if $k''$ is a finite Galois extension of $k''$ containing $k'$, then the connected components of $X \otimes_k k''$ are geometrically connected.*

*Corollary* (4.5.16). — *Let $X$ be a $k$-prescheme, $(X_\alpha)$ the family of its connected components, and $x_\alpha$ a point of $X_\alpha$ for each $\alpha$. Suppose that*

*(i) the family $(X_\alpha)$ is finite, and*

*(ii) for each $\alpha$, the separable algebraic closure $k'_\alpha$ of $k$ in $\mathbf{k}(x_\alpha)$ is a finite extension of $k$.* *Then the geometric number of connected components of $X$ is at most $\sum_\alpha [k'_\alpha : k]$, and there exists a finite separable extension $k''$ of $k$ such that all the connected components of $X \otimes_k k''$ are geometrically connected.*

*Corollary* (4.5.17). — *Suppose the $k$-prescheme $X$ contains a point $x$ such that the separable algebraic closure of $k$ in $\mathbf{k}(x)$ is finite over $k$. Then a necessary and sufficient condition for $X$ to be geometrically connected is that $X \otimes_k K$ is connected for every finite separable extension $k \subseteq K$.*

*Remark* (4.5.18). — We will see in §8 (8.4.5) that the conclusion of (4.5.17) also holds for every *quasi-compact* $k$-prescheme $X$.

*Proposition* (4.5.19). — *Let $X$ be a $k$-prescheme, $Z \subseteq X$ any subset, $k'$ an algebraically closed extension of $k$, $X' = X \otimes_k k'$, and $p \colon X' \to X$ the canonical projection. Suppose $Z' = p^{-1}(Z)$ is contained in a unique irreducible component $X'_0$ of $X'$. Then $X'_0 = p^{-1}(X_0)$, where $X_0 = p(X'_0)$ is an irreducible component of $X$ containing $Z$, and moreover $X_0$ is geometrically irreducible. If $X'_0$ is the unique irreducible component of $X'$ which meets $Z'$, then $X_0$ is the unique irreducible component of $X$ which meets $Z$.*

[We omit a Lemma (4.5.19.1) and diagram (4.5.19.2) that are used in the proof.]

*Remark* (4.5.20). — The hypotheses of (4.5.19) do not imply that $X_0 = p(X'_0)$ is the only irreducible component of $X$ containing $Z$. Example: let $k = \mathbb{R}$, $k' = \mathbb{C}$, $X_1 = \operatorname{Spec}(\mathbb{R}[S,T]/(S^2 + T^2 + 1))$, $X_2 = \mathbb{A}^1_{\mathbb{C}}$. The curve $X_1$ has no $\mathbb{R}$-rational points, so all its non-generic points have $\mathbf{k}(x) = \mathbb{C}$. Let $X$ be the union of $X_1$ and $X_2$, identified at a non-generic point $x_i$ in each scheme. [$X$ can be constructed as the affine scheme $\operatorname{Spec}(A_1 \times_{\mathbb{C}} A_2)$, where $X_i = \operatorname{Spec}(A_i)$ and the ring homomorphisms $A_i \to \mathbb{C}$ correspond to the morphisms $\operatorname{Spec}(\mathbb{C}) = \operatorname{Spec} \mathbf{k}(x_i) \to X_i$ for the points $x_i$ that we choose to identify.] If $x \in X_1 \cap X_2$ is the common point, then $p^{-1}(x)$ consists of two points $y', z'$, $p^{-1}(X_1)$ is irreducible, and $p^{-1}(X_2)$ is the disjoint union of two irreducible components $Y', Z'$ of $X'$ such that $y' \in Y'$ and $z' \in Z'$. Thus $p^{-1}(x)$ is contained in a unique irreducible component of $X'$.

*Proposition* (4.5.21). — *Let $k$ be a separably closed field and $k'$ the algebraic closure of $k$. For every $k$-prescheme $X$, the canonical projection $X \times_k k' \to X$ is universally a homeomorphism. In particular, the irreducible (resp. connected) components of $X$ are geometrically irreducible (resp. connected).*

## 4.6. **Geometrically reduced algebraic preschemes.**

*Proposition* (4.6.1). — *Let $k$ be a field, $X$ a $k$-prescheme, and $\Omega$ a perfect extension of $k$. The following are equivalent:*

*(a) For every reduced $k$-prescheme $S$, $X \times_k S$ is reduced.*
*(b) For every extension $k \subseteq K$, $X \otimes_k K$ is reduced.*
*(c) $X \otimes_k \Omega$ is reduced.*
*(d) For every finite radicial extension $k'$ of $k$, $X \times_k k'$ is reduced.*

*(e) $X$ is reduced, and for every irreducible component $X_\alpha$ of $X$, with generic point $x_\alpha$, $\mathbf{k}(x_\alpha)$ is a separable extension of $k$.*

*Definition* (4.6.2). — When conditions (a)-(e) in (4.6.1) hold, we say that $X$ is *separable*, or *geometrically reduced*, or *universally reduced* over $k$. We say that $X$ is *geometrically integral* over $k$ if $X \times_k K$ is integral for every extension $k \subseteq K$; by (4.6.1) this is the same as saying $X$ is separable and geometrically irreducible.

We say that a (commutative) $k$-algebra $A$ is *separable* if $\operatorname{Spec}(A)$ is separable over $k$; that is, for every extension $k \subseteq K$, the ring $A \times_k K$ is reduced. This coincides with the definition in Bourbaki, *Algebra* if $A$ has finite rank over $k$, but not in general.

*Corollary* (4.6.3). — *Let $X$ be an integral $k$-prescheme. For $X$ to be geometrically reduced (resp. geometrically integral) over $k$, it is necessary and sufficient that its field of rational functions $R(X)$ be separable (resp. separable and primary) over $k$.*

*Corollary* (4.6.4). — *Let $X$ be a reduced $k$-prescheme. Then $X \otimes_k k'$ is reduced for every separable extension $k'$ of $k$.*

*Proposition* (4.6.5). — *(i) Let $X$ be a $k$-prescheme, $k \subseteq K$ a field extension. Then $X$ is geometrically reduced over $k$ if and only if $X \otimes_k K$ is geometrically reduced over $K$.*

*(ii) If $X$ and $Y$ are $k$-preschemes geometrically reduced over $k$, then so is $X \times_k Y$.*

*Proposition* (4.6.6). — *Let $X$ be a $k$-prescheme of finite type. There exists a finite radicial extension $k'$ of $k$ such that $(X_{(k')})_{\mathrm{red}}$ is geometrically reduced over $k'$.*

*Corollary* (4.6.7). — *If $K$ is a finitely generated extension of $k$, there is a finite radicial extension $k'$ of $k$ such that the residue fields of the semi-local ring $K \otimes_k k'$ are separable over $k'$.*

*Corollary* (4.6.8). — *Let $X$ be a $k$-prescheme of finite type. There exists a finite extension $k'$ of $k$ such that $(X_{(k')})_{\mathrm{red}}$ is geometrically reduced over $k'$, the irreducible components of $X_{(k')}$ are geometrically irreducible, and the connected components of $X_{(k')}$ are geometrically connected.*

*Definition* (4.6.9). — Let $k$ be a field, $X$ a $k$-prescheme, and $x$ a point of $X$. We say that $X$ is *geometrically reduced* or *separable* (resp. *geometrically pointwise integral*) *at the point $x$* over $k$ if, for every extension $k \subseteq k'$ and every $x' \in X' = X \otimes_k k'$ lying over $x$, $X'$ is reduced (resp. integral) at $x'$, that is, $\mathcal{O}_{X',x'}$ is reduced (resp. integral). We say that $X$ is *geometrically pointwise integral* if it is geometrically pointwise integral at every point, that is, if for every extension $k \subseteq k'$, the local ring at every point of $X'$ is integral (in other words, $X'$ is *pointwise integral*).

Note that $X$ is geometrically reduced over $k$ if and only if it is geometrically reduced over $k$ at every point $x \in X$.

*Proposition* (4.6.10). — *Let $k$ be a field, $X$ a $k$-prescheme, $k'$ an extension of $k$, $x'$ a point of $X' = X \otimes_k k'$, $x$ its image in $X$. Then $X$ is geometrically reduced (resp. geometrically pointwise integral) at $x$ over $k$ if the same holds for $X'$ at $x'$ over $k'$.*

*Corollary* (4.6.11). — *Suppose $k$ is perfect (resp. algebraically closed). Then $X$ is geometrically reduced (resp. geometrically pointwise integral) at a point $x$ over $k$ if and only if $\mathcal{O}_{X,x}$ is reduced (resp. integral).*

*Proposition* (4.6.12). — *Let $k$ be a field, $X$ a $k$-prescheme, $x$ a point of $X$, $\Omega$ a perfect extension of $k$. The following conditions are equivalent.*

*(a) $X$ is geometrically reduced over $k$ at $x$, that is, for every extension $k \subseteq k'$ and every $x' \in X' = X \otimes_k k'$ lying over $x$, $\mathcal{O}_{X',x'}$ is reduced.*

*(b) The prescheme $X \otimes_k \Omega$ is reduced at some point lying over $x$.*

*(c) For every finite radicial extension $k'$ of $k$, $X' = X \otimes_k k'$ is reduced at the unique point $x'$ lying over $x$.*

*(d)* $\mathrm{Spec}(\mathcal{O}_{X,x})$ *is geometrically reduced over $k$.*

*(e) $\mathcal{O}_{X,x}$ is reduced, and for every irreducible component $Z$ of $X$ containing $x$, with generic point $z$, $\mathbf{k}(z)$ is a separable extension of $k$.*

*Corollary* (4.6.13). — *Under the hypotheses of (4.6.12), suppose further that $X$ is locally Noetherian. Then conditions (a)-(e) are also equivalent to:*

*(f) There exists an open neighborhood $U$ of $x$ which is geometrically reduced over $k$.*

(4.6.14). By (4.6.10) and (4.6.11), if $\Omega$ is an algebraically closed extension of $k$, then $X$ is geometrically pointwise integral at $x$ over $k$ if and only if $X \otimes_k \Omega$ is integral at some point lying over $x$.

When $X$ is locally Noetherian, (4.6.13) implies that the set of points $x \in X$ where $X$ is geometrically reduced is open, and this set is the largest open subset of $X$ that is geometrically reduced over $k$.

If $X$ is geometrically pointwise integral at $x$, then by (4.6.10), $X$ is necessarily integral at $x$, and $\mathbf{k}(z)$ is a separable extension of $k$, where $z$ is the generic point of the unique irreducible component of $X$ containing $x$. These conditions are not sufficient, however, as shown by example (4.5.12 (ii)). A sufficient but not necessary condition is that $X$ is geometrically reduced at $x$ and belongs to a unique irreducible component of $X$, which is geometrically irreducible. If $X$ is locally Noetherian, $x$ will then have a geometrically integral open neighborhood.

Recall that if a prescheme is locally Noetherian and pointwise integral, then it is locally integral (I, 6.1.13). If a $k$-prescheme $X$, locally of finite type over $k$, is geometrically pointwise integral over $k$, it follows that $X \otimes_k k'$ is locally integral for every extension $k \subseteq k'$; in this case we would say that $X$ is *geometrically locally integral.*

*Proposition* (4.6.15). — *(i) Let $X$ be a prescheme of finite type over a field $k$. Then $X$ is geometrically pointwise integral if and only if $X$ is geometrically reduced, and the geometric number of irreducible components of $X$ is equal to the geometric number of connected components.*

*(ii) Let $X$ be a prescheme locally of finite type over $k$. If $X$ is geometrically pointwise integral at $x$, then $x$ has a geometrically pointwise integral open neighborhood $U$. In other words, the set of points $x \in X$ where $X$ is geometrically pointwise integral is open.*

*Proposition* (4.6.16). — *Let $k$ be a field, $X$ a locally Noetherian $k$-prescheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-Module. The following conditions are equivalent:*

*(a) For every finitely generated extension $k \subseteq k'$, if we set $X' = X \otimes_k k'$, then $\mathcal{F}' = \mathcal{F} \otimes_k k'$ is a reduced $\mathcal{O}_{X'}$-Module (3.2.2).*

*(b) For every finite radicial extension $k \subseteq k'$, $\mathcal{F}'$ is a reduced $\mathcal{O}_{X'}$-Module.*

*(c) $\mathcal{F}$ is reduced, and if $\mathcal{J} \subseteq \mathcal{O}_X$ is the annihilator ideal of $\mathcal{F}$, the closed sub-prescheme of $X$ defined by $\mathcal{J}$ is geometrically reduced over $k$.*

*In addition, if $X$ is locally of finite type over $k$, the above conditions are also equivalent to*

*(d) For every extension $k'$ of $k$ (or for any one algebraically closed extension $k'$ of $k$), $\mathcal{F}'$ is a reduced $\mathcal{O}_{X'}$-Module.*

*Definition* (4.6.17). — If $\mathcal{F}$ satisfies the conditions in (4.6.16) we say that $\mathcal{F}$ is *geometrically reduced*, or *separable*, over $k$.

*Proposition* (4.6.18). — *Let $k$ be a field, $X$ a locally Noetherian $k$-prescheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-Module. The following conditions are equivalent:*

*(a) For every finitely generated extension $k \subseteq k'$, if we set $X' = X \otimes_k k'$, then $\mathcal{F}' = \mathcal{F} \otimes_k k'$ is an integral $\mathcal{O}_{X'}$-Module (3.2.4).*

*(b) For every finite extension $k \subseteq k'$, $\mathcal{F}'$ is an integral $\mathcal{O}_{X'}$-Module.*

*(c) $\mathcal{F}$ is reduced (or integral), and if $\mathcal{J} \subseteq \mathcal{O}_X$ is the annihilator ideal of $\mathcal{F}$, the closed sub-prescheme of $X$ defined by $\mathcal{J}$ is geometrically integral over $k$.*

*In addition, if $X$ is locally of finite type over $k$, the above conditions are also equivalent to*

*(d) For every extension $k'$ of $k$ (or for any one algebraically closed extension $k'$ of $k$), $\mathcal{F}'$ is an integral $\mathcal{O}_{X'}$-Module.*

*Definition* (4.6.19). — If $\mathcal{F}$ satisfies the conditions in (4.6.18), we say that $\mathcal{F}$ is *geometrically integral* over $k$.

*Proposition* (4.6.20). — *Let $k$ be a field, $X$ a locally Noetherian $k$-prescheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$ module. Let $k \subseteq K$ be an extension such that $X \otimes_k K$ is locally Noetherian. Then, if $\mathcal{F}$ is geometrically reduced (resp. geometrically integral) over $k$, $\mathcal{F} \otimes_k K$ has the same property over $K$.*

*Proposition* (4.6.21). — *Let $X, Y$ be locally Noetherian $k$-preschemes such that $X \times_k Y$ is locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-Module and $\mathcal{G}$ a coherent $\mathcal{O}_Y$-Module.*

*(i) If $\mathcal{F}$ is geometrically reduced (resp. geometrically integral) and $\mathcal{G}$ is reduced (resp. integral), then $\mathcal{F} \otimes_k \mathcal{G}$ is reduced (resp. integral).*

*(ii) If $\mathcal{F}$ and $\mathcal{G}$ are geometrically reduced (resp. geometrically integral), then so is $\mathcal{F} \otimes_k \mathcal{G}$.*

*Definition* (4.6.22). — Let $k$ be a field, $X$ a locally Noetherian $k$-prescheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-Module. We say that $\mathcal{F}$ is *geometrically reduced*, or *separable* (resp. *geometrically pointwise integral*) at a point $x \in X$ if, for every finite radicial extension (resp. finite extension) $k \subseteq k'$, $\mathcal{F} \otimes_k k'$ is reduced (resp. integral) at every point $x' \in X \otimes_k k'$ above $x$.

If $\mathcal{F}$ is reduced at $x$, then $\mathcal{F}|U$ is reduced, for some open neighborhood $U$ of $x$ (3.2.2). As in the [omitted] proof of (4.6.16), $\mathcal{F}$ is geometrically reduced at $x$ if and only if $\mathcal{F}$ is reduced at $x$ and the closed subscheme $Y$ defined by the annihilator ideal $\mathcal{J} \subseteq \mathcal{O}_X$ of $\mathcal{F}$ is geometrically reduced at $x$. It follows that $\mathcal{F}$ is geometrically reduced at $x$ if and only if $\mathcal{F}|U$ is geometrically reduced, for some open neighborhood $U$ of $x$. If $X$ is locally of finite type over $k$, we also have that $\mathcal{F}$ is pointwise geometrically integral at $x$ if and only if $\mathcal{F}$ is reduced at $X$ and the subscheme $Y$ is geometrically pointwise integral at $x$.