## Synopsis of material from EGA Chapter IV, §10.1–10.4

### 10. Jacobson preschemes

**10.1. Very dense subsets of a topological space.**

(10.1.1). A subset $T$ of a topological space $X$ is *quasi-constructible* if $T$ is a finite union of locally closed subsets. $T$ is *locally quasi-constructible* if every $x \in X$ has an open neighborhood $V$ such that $T \cap V$ is quasi-constructible in $V$. The two notions are equivalent if $X$ is quasi-compact. Let $\mathfrak{Qc}(X)$, $\mathfrak{Lqc}(X)$ denote the set of (locally) quasi-constructible subsets. Then $\mathfrak{Qc}(X)$ and $\mathfrak{Lqc}(X)$ are closed under finite intersections, unions, and complements, and preimages via continuous maps. Let $\mathfrak{O}(X)$ denote the set of open subsets of $X$, $\mathfrak{Cl}(X)$ the set of closed subsets.

*Proposition* (10.1.2). — *Let $X_0$ be a subspace of $X$. The following are equivalent.*

*(a) For every non-empty locally closed $Z \subseteq X$, $Z \cap X_0 \neq \emptyset$.*

*(a′) For every closed $Z \subseteq X$, $Z = \overline{Z \cap X}$.*

*(b) For every non-empty locally quasi-constructible $Z \subseteq X$, $Z \cap X_0 \neq \emptyset$.*

*(b′) For every locally quasi-constructible $Z \subseteq X$, $Z \subseteq \overline{Z \cap X_0}$, that is, $Z \cap X_0$ is dense in $Z$.*

*(c) $U \mapsto U \cap X_0$ from $\mathfrak{O}(X)$ to $\mathfrak{O}(X_0)$ is injective (hence bijective).*

*(c′) $Z \mapsto Z \cap X_0$ from $\mathfrak{Cl}(X)$ to $\mathfrak{Cl}(X_0)$ is injective (hence bijective).*

*(c″) $Z \mapsto Z \cap X_0$ from $\mathfrak{Qc}(X)$ to $\mathfrak{Qc}(X_0)$ is injective (hence bijective).*

*(c‴) $Z \mapsto Z \cap X_0$ from $\mathfrak{Lqc}(X)$ to $\mathfrak{Lqc}(X_0)$ is injective (which implies that it is bijective).*

*Definition* (10.1.3). — When the conditions in (10.1.2) hold, we say that $X_0$ is *very dense* in $X$.

*Corollary* (10.1.4). — *If $X_0$ is very dense in $X$, and $U \subseteq X$ is open, then $U \cap X_0$ is very dense in $U$. Conversely, if $X = \bigcup_\alpha U_\alpha$ is an open covering such that $U_\alpha \cap X_0$ is very dense in $U_\alpha$ for each $\alpha$, then $X_0$ is very dense in $X$.*

**10.2. Quasi-homeomorphisms.**

*Proposition* (10.2.1). — *Let $f\colon X_0 \to X$ be a continuous map. The following are equivalent.*

*(a) $U \mapsto f^{-1}(U)$ from $\mathfrak{O}(X)$ to $\mathfrak{O}(X_0)$ is bijective .*

*(a′) $Z \mapsto f^{-1}(Z)$ from $\mathfrak{Cl}(X)$ to $\mathfrak{Cl}(X_0)$ is bijective .*

*(b) The topology on $X_0$ is the inverse image of that on $X$, and $f(X_0)$ is very dense in $X$.*

*(c) The functor $f^{-1}$ from sheaves on $X$ to sheaves on $X_0$ is an equivalence of categories (this holds both for sheaves of sets and for sheaves of abelian groups).*

*Definition* (10.2.2). — A map $f$ satisfying the conditions in (10.2.1) is a *quasi-homeomorphism*.
In particular, by (10.2.1, b), a subspace $X_0 \subseteq X$ is very dense if and only if the inclusion $X_0 \hookrightarrow X$ is a quasi-homeomorphism.

*Corollary* (10.2.3). — *The composite of two quasi-homeomorphisms is a quasi-homeomorphism.*

*Corollary* (10.2.4). — *If $f\colon X \to Y$ is a quasi-homeomorphism, $Y' \subseteq Y$ is locally quasi-constructible, and $X' = f^{-1}(Y')$, then the restriction $f' = (f|X')\colon X' \to Y'$ is a quasi-homeomorphism.*

*Corollary* (10.2.5). — *Let $f\colon X \to Y$ be a continuous map, $Y = \bigcup_\alpha V_\alpha$ an open covering. If the restriction $f^{-1}(V_\alpha) \to V_\alpha$ of $f$ is a quasi-homeomorphism for all $\alpha$, then $f$ is a quasi-homeomorphism.*

*Corollary* (10.2.6). — *Let $f\colon X \to Y$ be a quasi-homeomorphism, $Y' \subseteq Y$ locally quasi-constructible, $X' = f^{-1}(Y')$. Then $Y'$ is quasi-compact (resp. Noetherian, retro-compact) iff $X'$ is.*

*Proposition* (10.2.7). — *Let $f\colon X \to Y$ be a quasi-homeomorphism. Then the map $Z \mapsto f^{-1}(Z)$ from subsets of $Y$ to subsets of $X$ induces bijections between the open, closed, locally closed, quasi-constructible, locally quasi-constructible, constructible, and locally constructible subsets of $X$ and $Y$.*

*Remarks* (10.2.8). — (i) If $f\colon X \to Y$ is a quasi-homeomorphism then for any sheaf of abelian groups $\mathcal{F}$ on $Y$, the canonical functorial map

(10.2.8.1) $$\Gamma(Y, \mathcal{F}) \to \Gamma(X, f^{-1}(\mathcal{F}))$$

is an isomorphism. Since $f^{-1}$ is exact, it follows that the canonical maps in cohomology

$$H^i(Y, \mathcal{F}) \to H^i(X, f^{-1}(\mathcal{F}))$$

are isomorphisms.

(ii) If $f$ is a quasi-homeomorphism then $f^{-1}$ gives an equivalence between the categories of sheaves of rings on $X$ and on $Y$. If $f = (\psi, \theta)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, $\psi$ is a quasi-homeomorphism, and $\theta^\sharp$ is an isomorphism, then $f^* = f^{-1}$ gives an equivalence of categories between sheaves of $\mathcal{O}_Y$ modules and sheaves of $\mathcal{O}_X$ modules. This extends to isomorphisms of Ext functors, and more generally to equivalences between all the usual functorial constructions involving sheaves and cohomology on the two spaces $X$ and $Y$.

The 1971 revised edition of EGA I by Grothendieck and Dieudonné, which also includes material from EGA IV, §10, adds to the above the following definition and results.

*Definition.* A topological space $X$ is *sober* if every irreducible closed subset $Z \subseteq X$ has a unique generic point, that is, a point $z$ such that $Z = \overline{\{z\}}$ (0, 2.1.2).

Every prescheme $X$ is sober (I, 2.1.5).

For any space $X$, let $X^+$ denote the set of irreducible closed subsets of $X$. If $V \subseteq X$ is closed, then $V^+$ is a subset of $X^+$. The correspondence $V \mapsto V^+$ preserves finite unions and arbitrary intersections, making the subsets $V^+$ the closed subsets of a topology on $X^+$.

The map $j\colon X \to X^+$ defined by $j(z) = \overline{\{z\}}$ is continuous, and $j^{-1}(V^+) = V$. It follows that $V \to V^+$ is a bijection from closed subsets of $X$ to closed subsets of $X^+$ and $j^{-1}$ induces its inverse. Hence $j$ is a quasi-homeomorphism.

The space $X^+$ is sober. Its irreducible closed subsets are exactly the sets $Z^+$ for $Z \in X^+$, and we have $Z^+ = \overline{\{Z\}}$.

Given a continuous map $f\colon X \to Y$, there is a unique continuous map $f^+\colon X^+ \to Y^+$ such that $f^+ j_X = j_Y f$. This makes $(-)^+$ a functor from topological spaces to sober spaces. Every continuous map $f\colon X \to Y$, where $Y$ is sober, factors uniquely through $j\colon X \to X^+$. This implies that $(-)^+$ is left adjoint to the inclusion of sober spaces into topological spaces.

Every quasi-homeomorphism between sober spaces is a homeomorphism. It follows that a continuous map $f\colon X \to Y$ is a quasi-homeomorphism if and only if $f^+$ is a homeomorphism. One can therefore view sober spaces as canonical representatives of topological spaces up to quasi-homeomorphism.

## 10.3. **Jacobson spaces.**

*Definition* (10.3.1). — A topological space $X$ is *Jacobson* if the set of closed points $X_0$ of $X$ is very dense in $X$; that is, if $X_0 \hookrightarrow X$ is a quasi-homeomorphism.

*Proposition* (10.3.2). — *Let $X$ be Jacobson, $Z \subseteq X$ locally quasi-constructible. Then the subspace $Z$ is Jacobson, and a point $z \in Z$ is closed in $Z$ iff it is closed in $X$.*

*Proposition* (10.3.3). — *Let $X = \bigcup_\alpha U_\alpha$ be an open covering. Then $X$ is Jacobson iff every $U_\alpha$ is Jacobson.*

## 10.4. **Jacobson preschemes and Jacobson rings.**

*Definition* (10.4.1). — A prescheme $X$ is *Jacobson* if its underlying topological space is Jacobson. A ring $A$ is *Jacobson* if $\operatorname{Spec}(A)$ is Jacobson.

According to this definition, $A$ is Jacobson if and only if every radical ideal of $A$ is an intersection of maximal ideals; if and only if every prime ideal of $A$ is an intersection of maximal ideals (the latter is the usual definition of a Jacobson ring).

*Proposition* (10.4.2). — *Let $X = \bigcup_\alpha U_\alpha$ be an open affine covering of the prescheme $X$. Then $X$ is Jacobson iff each ring $\mathcal{O}_X(U_\alpha)$ is Jacobson.*

(10.4.3). Examples: a discrete space is Jacobson, hence an Artinian ring is Jacobson. A principal ideal domain with infinitely many maximal ideals (such as $\mathbb{Z}$) is Jacobson. A Noetherian local ring is Jacobson iff its maximal ideal is its only prime ideal; that is, iff it is Artinian. By (10.3.2), any sub-prescheme of a Jacobson scheme is Jacobson.

*Proposition* (10.4.4). — *Let $B$ be an integral domain. The following are equivalent.*

*(a) There exists $f \neq 0$ in $B$ such that $B_f$ is a field.*

*(b) The field of fractions of $B$ is a finitely generated $B$-algebra.*

*(c) There exists a field $K$ containing $B$, which is a finitely generated $B$-algebra.*

*(d) The generic point of* $\operatorname{Spec}(B)$ *is isolated (i.e., the set consisting of only that point is open).*

(d) $\Leftrightarrow$ (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) are easy. The significant point is that (c) implies the others, which is a version of Hilbert's Nullstellensatz.

*Proposition* (10.4.5). — *Given a ring $A$, the following are equivalent.*

*(a) $A$ is Jacobson.*

*(b) For every non-maximal prime ideal $\mathfrak{p} \subseteq A$ and every $f \neq 0$ in $B = A/\mathfrak{p}$, $B_f$ is not a field.*

*(b′) Every finitely generated $A$-algebra $K$ which is a field, is finite over $A$ (i.e., finitely generated as an $A$-module; thus a finite algebraic extension of $A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal).*

*Corollary* (10.4.6). — *Every algebra $B$ of finite type over a Jacobson ring $A$ is Jacobson. Moreover, the preimage in $A$ of any maximal ideal of $B$ is maximal. In particular, any finitely generated algebra over $\mathbb{Z}$ or a field is Jacobson.*

*Corollary* (10.4.7). — *If $X$ is a Jacobson prescheme and $f\colon Y \to X$ is a morphism locally of finite type, then $Y$ is Jacobson, and $f$ maps every closed point of $X$ to a closed point of $Y$. [Moreover, if $f(x) = y$, then $k(x)$ is a finite algebraic extension of $k(y)$.]*

*Corollary* (10.4.8). — *If $X$ is locally of finite type over an algebraically closed field $k$, then the $k$-rational points of $X$ are very dense in $X$.*

Indeed, the $k$-rational points are the closed points, by (I, 6.4.2), and $X$ is Jacobson.

(10.4.9–11). A number of questions in algebraic geometry can be reduced to the case of a finitely generated algebra over $\mathbb{Z}$ or a field, so the fact that such rings are Jacobson is particularly important. EGA gives two applications, of which the second is the following.

*Proposition*: *Let $X$ be an $S$-prescheme of finite type. Then any universally injective $S$-morphism $g\colon X \to X$ is bijective.*

[The morphism $g$ is *universally injective* if it induces an injection $X(K) \to X(K)$ for every field $K$.]

In fact, it is shown in (IV, 17.9.7) that under the hypotheses of the Proposition, $g$ is an isomorphism.