Synopsis of material from EGA Chapter II, §5

5. Quasi-affine, quasi-projective, proper and projective morphisms

5.1. **Quasi-affine morphisms.**

Definition (5.1.1). — A scheme is *quasi-affine* if it is isomorphic to a quasi-compact open subscheme of an affine scheme. A morphism $f\colon X \to Y$ is *quasi-affine* if there exists a covering of Y by open affines U_α such that $f^{-1}(U_\alpha)$ is quasi-affine.

Every quasi-affine morphism is separated and quasi-compact. An affine morphism is quasi-affine.

Recall that for any prescheme X, putting $A = \Gamma(X, \mathcal{O}X)$, there is a canonical morphism $X \to \operatorname{Spec}(A)$.

Proposition (5.1.2). — *Let X be quasi-compact or topologically Noetherian [or more generally, quasi-compact and quasi-separated (IV, 1.7.16)], $A = \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

(a) X is quasi-affine.
(b) The canonical morphism $u\colon X \to \operatorname{Spec}(A)$ is an open immersion.
(b′) The canonical morphism u is a homeomorphism of X onto its image in $\operatorname{Spec}(A)$.
(c) $\mathcal{O}_X$ is very ample for u.
(c′) $\mathcal{O}_X$ is ample.
(d) The X_f for $f \in A$ form a base of the topology on X.
(d′) Those X_f which are affine cover X.
(e) Every quasi-coherent $\mathcal{O}_X$-module is generated by its global sections.
(e′) Every quasi-coherent ideal sheaf of finite type in $\mathcal{O}_X$ is generated by its global sections.

Observe that when these conditions hold, the affines X_f form a base of the topology, and u is dominant.

Corollary (5.1.3). — *If X is quasi-compact, and $v\colon X \to Y$ is a morphism to an affine scheme Y, which is a homeomorphism of X onto an open subspace of Y, then X is quasi-affine.*

Corollary (5.1.4). — *If X is quasi-affine, every invertible sheaf is very ample (relative to the canonical morphism), and hence ample.*

Corollary (5.1.5). — *Let X be a quasi-compact prescheme. If there is an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L}$ and $\mathcal{L}^{-1}$ are both ample, then X is quasi-affine.*

Proposition (5.1.6). — *Let $f\colon X \to Y$ be a quasi-compact morphism. The following are equivalent:*

(a) f is quasi-affine.
(b) $\mathcal{A} = f_*(\mathcal{O}_X)$ is quasi-coherent, and the canonical Y-morphism $u\colon X \to \operatorname{Spec}(\mathcal{A})$ is an open immersion.
(b′) Like (b), but only assuming u is a homeomorphism onto its image.
(c) $\mathcal{O}_X$ is very ample relative to f.

(c′) $\mathcal{O}_X$ is ample relative to f.

(d) f is separated, and for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, $\sigma\colon f^*(f_*(\mathcal{F})) \to \mathcal{F}$ is surjective.

Moreover, if f is quasi-affine, then every invertible sheaf on X is very ample relative to f.

Corollary (5.1.7). — *Let $f\colon X \to Y$ be quasi-affine. For every open $U \subseteq Y$, the restriction $f^{-1}(U) \to U$ of f is quasi-affine.*

Corollary (5.1.8). — *Let $f\colon X \to Y$ be quasi-compact, Y affine. Then f is quasi-affine iff X is a quasi-affine scheme.*

Corollary (5.1.9). — *Let Y be a quasi-compact scheme, or a topologically Noetherian prescheme [or more generally, a quasi-compact and quasi-separated prescheme], $f\colon X \to Y$ a morphism of finite type. If f is quasi-affine, there is a quasi-coherent sub-$\mathcal{O}_Y$-algebra $\mathcal{B} \subseteq \mathcal{A}(X) = f_*(\mathcal{O}_X)$ of* finite type *such that the morphism $X \to \mathrm{Spec}(\mathcal{B})$ corresponding to $\mathcal{B} \hookrightarrow \mathcal{A}(X)$ (1.2.7) is an immersion. Moreover, every quasi-coherent subalgebra $\mathcal{B}' \subseteq \mathcal{A}(X)$ containing $\mathcal{B}$ has the same property.*

[By (3.1.7), this is special case of (3.8.4).]

Proposition (5.1.10). — *(i) Any quasi-compact morphism $X \to Y$ which is a homeomorphism onto its image—in particluar, any closed immersion—is quasi-affine.*

(ii) The composite of of two quasi-affine morphisms is quasi-affine.

(iii) If $f\colon X \to Y$ is a quasi-affine S-morphism, any base extension $f_{(S')}$ is quasi-affine.

(iv) If f and g are quasi-affine S-morphisms, so is $f \times_S g$.

(v) If $g \circ f$ is quasi-affine, and if g is separated or if X is topologically locally Noetherian, then f is quasi-affine.

(vi) If f is quasi-affine, then so is f_{red}.

Remark (5.1.11). — Given $f\colon X \to Y$, $g\colon Y \to Z$, (v) also holds under the alternative assumption that $X \times_Z Y$ is locally Noetherian.

Proposition (5.1.12). — *Let $f\colon X \to Y$ be quasi-compact, $g\colon X' \to X$ quasi-affine. If $\mathcal{L}$ is ample for f, then $g^*(\mathcal{L})$ is ample for $f \circ g$.*

5.2. **Serre's criterion.**

Theorem (5.2.1). — (Serre's criterion) *Let X be a quasi-compact scheme or a topologically Noetherian prescheme. The following are equivalent:*

(a) X is affine.

(b) There exist elements $f_\alpha \in A = \Gamma(X, \mathcal{O}_X)$ such that the X_{f_α} are affine and the f_α generate the unit ideal in A.

(c) $\Gamma(X, -)$ is an exact functor on the category of quasi-coherent $\mathcal{O}_X$-modules.

(c′) $\Gamma(X, -)$ is exact on sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ such that $\mathcal{F}$ is a sub-$\mathcal{O}_X$-module of a finite-rank free sheaf $\mathcal{O}_X^n$.

(d) $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.

(d′) Property (d) holds for every quasi-coherent ideal sheaf $\mathcal{F}$.

(a)⇔(b) follows from (4.5.2).

(a)⇒(c)⇒(c′) by (I, 1.3.11). Given (c′), one shows first that sets X_f form a neighborhood base at each closed point $x \in X$, by letting $x \in U$ be a neighborhood, $\mathcal{I}$ the ideal of $X \setminus U$, $\mathcal{I}'$ the ideal of $\{x\} \cap (X \setminus U)$, and applying (c′) to $0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0$. Using quasi-compactness and (0, 2.1.3), we get a covering of X by affines X_{f_i}. Applying (c′) to the surjection $\mathcal{O}_X^n \Rightarrow \mathcal{O}$ defined by the f_i's shows that they generate the unit ideal, giving (b).

(a)⇒(d)⇒(d′) by the cohomology vanishing theorem for affine schemes, and (d′)⇒(c′) by some applications of the long exact sequence of cohomology.

Corollary (5.2.2). — *Let $f\colon X \to Y$ be a quasi-compact morphism, and assume X is separated or topologically locally Noetherian [or assume f quasi-separated (IV, 1.7.18)]. The following are equivalent:*

(a) f is an affine morphism.

(b) f_* is exact on quasi-coherent $\mathcal{O}_X$-modules

(c) $R^1 f_*(\mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.

(c′) Property (c) holds for quasi-coherent ideal sheaves $\mathcal{F}$.

Corollary (5.2.3). — *If $f\colon X \to Y$ is an affine morphism, then for all quasi-coherent $\mathcal{F}$ on X, we have $H^i(Y, f_*(\mathcal{F})) \cong H^i(X, \mathcal{F})$.*

[Stated in EGA for $i = 1$, but it holds for all i because $Rf_* = f_*$ and $R\Gamma(X, -) = R\Gamma(Y, -) \circ Rf_*$.]

5.3. **Quasi-projective morphisms.**

Definition (5.3.1). — A morphism $f\colon X \to Y$ is *quasi-projective* (or X is a *quasi-projective scheme over* Y) if f is of finite type and there exists an invertible $\mathcal{O}_X$-module ample relative to f.

Warning: this condition is *not* local on Y, even if X and Y are algebraic schemes over an algebraicallly closed field.

A quasi-projective morphism is necessarily separated. If Y is quasi-compact, one can replace “ample” with “very ample” (4.6.2 and 4.6.11).

Proposition (5.3.2). — *Let Y be a quasi-compact scheme or a topologically Noetherian prescheme [or more generallly, a quasi-compact and quasi-separated prescheme]. The following are equivalent:*

(a) X is a quasi-projective Y-scheme.

(b) X is of finite type over Y, and there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ of finite type such that X is Y-isomorphic to a sub-prescheme of $\mathbb{P}(\mathcal{E})$.

(c) X is of finite type over Y, and there is a graded quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{S}$ such that $\mathcal{S}_1$ is of finite type and generates $\mathcal{S}$, and X is Y-isomorphic to a dense open subscheme of $\operatorname{Proj}(\mathcal{S})$.

Corollary (5.3.3). — *Let Y be a quasi-compact scheme with an ample invertible sheaf $\mathcal{L}$. A Y-scheme X to is quasi-projective iff X is of finite type over Y and Y-isomorphic to a sub-Y-scheme of some $\mathbb{P}^r_Y$.*

Proposition (5.3.4). — *(i) Every quasi-affine morphism of finite type—in particular, every quasi-compact immersion—is quasi-projective.*

(ii) The composite of two quasi-projective morphisms is quasi-projective.

(iii) Every base extension of a quasi-projective morphism is quasi-projective.

(iv) The product of two quasi-projectivce S-morphisms is quasi-projective.

(v) If $g \circ f$ is quasi-projective, and g is separated or X is locally Noetherian, then f is quasi-projective.

(vi) If f is quasi-projective, so is f_{red}.

Remark (5.3.5). — It is possible for f_{red} but not f to be quasi-projective.

Corollary (5.3.6). — *If X and X' are quasi-projective Y-schemes, then so is $X \bigsqcup X'$.*

5.4. **Proper and universally closed morphisms.**

Definition (5.4.1). — A morphism $f\colon X \to Y$ is *proper* if it satisfies the two conditions:

(a) f is separated and of finite type.

(b) For every $Y' \to Y$, the base extension $f_{(Y')}\colon X \times_Y Y' \to Y'$ is closed (I, 2.2.6).

We also say that X is a *proper Y-scheme.*

Properness is a local condition on Y. Clearly, a proper morphism is closed.

Proposition (5.4.2). — *(i) Every closed immersion is proper.*

(ii) The composite of two proper morphisms is proper.

(iii) Every base extension of a proper morphism is proper.

(iv) The product of two proper S-morphisms is proper.

Corollary (5.4.3). — *Given $f\colon X \to Y$, $g\colon Y \to Z$, suppose $g \circ f$ is proper.*

(i) If g is separated, then f is proper.

(ii) If g is separated and of finite type, and f is surjective, then g is proper.

Corollary (5.4.4). — *If X is proper over Y, and $\mathcal{S}$ is a graded quasi-coherent $\mathcal{O}_Y$-algebra, then every Y-morphism $X \to \operatorname{Proj}(\mathcal{S})$ is proper, hence closed.*

Corollary (5.4.5). — *Let $f\colon X \to Y$ be a separated morphism of finite type. Let X_i (resp. Y_i), $i = 1, \ldots, n$ be closed subschemes of X (resp. Y), $j_i\colon X_i \hookrightarrow X$, $h_i\colon Y_i \hookrightarrow Y$ the inclusions. Suppose the underlying space of X is the union of the X_i's and that for each i there is a morphism $f_i\colon X_i \to Y_i$ such that $h_i f_i = f j_i$. Then f is proper if and only if each f_i is proper.*

Corollary (5.4.6). — *Let f be a separated morphism of finite type. Then f is proper if and only if f_{red} is proper.*

(5.4.7). If $f\colon X \to Y$ is a separated morphism of finite type between Noetherian preschemes, we can take X_i in (5.4.5) to be the induced reduced subschemes of the irreducible components of X, and Y_i the closures of their images. Then the verification that f is proper reduces to that for each f_i, which is a dominant morphism of integral preschemes.

Corollary (5.4.8). — *Let $f\colon X \to Y$ be an S-morphism of S-schemes of finite type. Then f is proper if and only if $f_{(S')}$ is closed for every S-scheme S'.*

Remark (5.4.9). — A morphism satifying (5.4.1 (b)) is called *universally closed.* In (5.4.2) through (5.4.8) one can replace "proper" with "universally closed" and omit the finiteness hypotheses.

(5.4.10). Let $f\colon X \to Y$ be a morphism of finite type. A closed subset $Z \subseteq X$ is *proper over* Y, or *proper for* f, if the restriction of f to a closed subscheme of X with underlying space Z is proper. This doesn't depend on the choice of closed subscheme, by (5.4.6).

If $g\colon X' \to X$ is proper, then $g^{-1}(Z)$ is proper. If $u\colon X \to X''$ is any Y-morphism, where X'' is a Y-*scheme* of finite type, then $u(Z)$ is proper.

In particular, if Z is a Y-proper subset of X, then $Z \cap X'$ is a Y-proper subset of X', for any closed subscheme $X' \subseteq X$; and if X is a sub-prescheme of a Y-scheme X'' of finite type, then Z is a Y-proper subset of X'' (hence closed).

5.5. **Projective morphisms.**

Proposition (5.5.1). — *Let X be a Y-prescheme. The following are equivalent:*

(a) X is Y-isomorphic to a closed subscheme of $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type.

(b) X is Y-isomorphic to $\mathrm{Proj}(\mathcal{S})$*, where $\mathcal{S}$ is a graded quasi-coherent $\mathcal{O}_Y$-algebra such that $\mathcal{S}_1$ is of finite type and generates $\mathcal{S}$.*

Definition (5.5.2). — If the conditions in (5.5.1) hold, X is *projective* over Y. A morphism $f\colon X \to Y$ is *projective* if it makes X a projective Y-scheme.

Clearly if $f\colon X \to Y$ is projective, there exists an $\mathcal{O}_X$-module very ample for f.

Theorem (5.5.3). — *(i) Every projective morphism is quasi-projective and proper.*

(ii) Conversely, if Y is a quasi-compact scheme, or a topologically Noetherian prescheme [or more generally, if Y is quasi-compact and quasi-separated (IV, 1.7.19)], then every proper quasi-projective morphism is projective.

(ii) follows from (5.3.2) and (5.4.4).

Projective morphisms are preserved by base extension, so for (i) we must show that projective morphisms are closed. This reduces to the case $Y = \mathrm{Spec}(A)$, $X = \mathrm{Proj}(S)$, where S is a graded A-algebra generated by finitely many elements of S_1. Using $f^{-1}(y) \cong \mathrm{Proj}(S \otimes_A k(y))$ and Nakayama's lemma, one proves that $f(X) = \bigcap_n \mathrm{Supp}(S_n)$, which is closed. Since any closed $X' \subseteq X$ is again projective over Y, this shows that f is a closed morphism.

Remarks (5.5.4). — (i) Suppose 1^o $f\colon X \to Y$ is proper, 2^o there exists an $\mathcal{O}_X$-module $\mathcal{L}$ very ample for f, and 3^o the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E} = f_*(\mathcal{L})$ is of finite type. Then f is projective, by (4.4.4) and (5.4.4). In volume III, §3, we shall see that if Y is locally Noetherian, then 1^o and 2^o imply 3^o, hence they characterize projective morphisms. If Y

is quasi-compact, one can replace 2^o by the existence of an $\mathcal{O}_X$-module ample relative to f (4.6.11).

(ii) Let Y be a quasi-compact scheme possessing an ample $\mathcal{O}_Y$-module. Then a Y-scheme X is projective iff it is Y-isomorphic to a closed subscheme of $\mathbb{P}^r_Y$ for some r, by (5.3.3), (5.4.4) and (5.5.3).

(iii) The proof of (5.5.3) shows that $\mathbb{P}^r_Y \to Y$ is always surjective.

(iv) There exist proper morphisms which are not quasi-projective. [The earliest counterexamples are due to Nagata. A simple class of more recent counterexamples is given by toric varieties associated to non-polyhedral fans.]

Proposition (5.5.5). — *(i) Every closed immersion is projective.*

(ii) If $f\colon X \to Y$ and $g\colon Y \to Z$ are projective, and Z is a quasi-compact scheme or a topologically Noetherian prescheme [or Z is quasi-compact and quasi-separated], then $g \circ f$ is projective.

(iii) Every base extension of a projective morphism is projective.

(iv) The product of two projectivce S-morphisms is projective.

(v) If $g \circ f$ is projective and g is separated, then f is projective.

(vi) If f is projective, so is f_{red}.

Proposition (5.5.6). — *If X, X' are projective Y-schemes, then so is $X \sqcup X'$.*

Proposition (5.5.7). — *Let X be a projective Y-scheme, $\mathcal{L}$ a Y-ample $\mathcal{O}_X$-module. For every global section f of $\mathcal{L}$, X_f is affine over Y.*

Corollary (5.5.8). — *Let X be a prescheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. For every global section f of $\mathcal{L}$, X_f is affine over X (in particular, X_f is affine if X is affine).*

This can also be shown directly, without using (5.5.7).

5.6. **Chow's Lemma.**

Theorem (5.6.1). — *(Chow's lemma) Let X be an S-scheme of finite type. Suppose that either*

(a) S is Noetherian; or

(b) S is a quasi-compact scheme, and X has finitely many irreducible components.

Then:

(i) There exists a projective and surjective *S-morphism $f\colon X' \to X$, where X' is a* quasi-projective *S-scheme.*

(ii) X' and f can be chosen so that there is an open $U \subseteq X$ such that $U' = f^{-1}(U)$ is dense in X', and f restricts to an isomorphism $U' \to U$.

(iii) If X is reduced (resp. irreducible, integral), one can take X' to have the same property.

Corollary (5.6.2). — *Under the hypotheses (a) or (b) of (5.6.1), for X to be proper over S it is necessary and sufficient that there exist a projective S-scheme X' and a surjective S-morphism $f\colon X' \to X$ (which is then projective by (5.5.5, (v))). Moreover, f can be chosen such that some dense open $U \subseteq X$ is the isomorphic image of $U' = f^{-1}(U)$, with U' dense* *in* X'*. If* X *is irreducible (resp. reduced), then* X' *can be chosen to have the same property; thus if* X *and* X' *are irreducible,* f *is a birational morphism.*

Corollary (5.6.3). — *Let* $f\colon X \to S$ *make* X *a scheme of finite type over a locally Noetherian prescheme* S*. Then* X *is proper over* S *if and only if for every morphism* of finite type $S' \to S$*, the base extension* $f_{(S')}$ *is closed. It even suffices that this hold for* $S' = \mathbb{A}^n_S = S \otimes_{\mathbb{Z}} \mathbb{Z}[t_1, \dots, t_n]$*, for all* n*.*