Synopsis of material from EGA Chapter II, §2.5–2.9

2.5. **Sheaf associated to a graded module.**

(2.5.1). If M is a graded S module, then $M_{(f)}$ is an $S_{(f)}$ module, giving a quasi-coherent sheaf $\widetilde{M_{(f)}}$ on $\mathrm{Spec}(S_{(f)}) = D_+(f) \subseteq \mathrm{Proj}(S)$ (I, 1.3.4).

Proposition (2.5.2). — *Given a graded S module M, there is a unique quasi-coherent sheaf $\widetilde{M}$ of $\mathcal{O}_X$ modules on $X = \mathrm{Proj}(S)$ such that $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ for every homogeneous $f \in S_+$, with restriction from $D_+(f)$ to $D_+(fg)$ given by the canonical homomorphism $M_{(f)} \to M_{(fg)}$.*

Definition (2.5.3). — *$\widetilde{M}$ in (2.5.2) is the* sheaf associated to *the graded S module M.*

Proposition (2.5.4). — *$M \mapsto \widetilde{M}$ is an exact functor which commutes with inductive limits and arbitrary direct sums.*

Proposition (2.5.5). — *For all $\mathfrak{p} \in \mathrm{Proj}(S)$, we have $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$.*

Proposition (2.5.6). — *Suppose that for every $z \in M$ and every homogeneous $f \in S_+$, some power of f annihilates z. Then $\widetilde{M} = 0$. If S_1 generates S as an S_0-algebra, the converse holds.*

Proposition (2.5.7). — *Let $f \in S_d$, $d > 0$. For every integer n, the sheaf $S(nd)\tilde{}\,|D_+(f)$ is isomorphic to $\mathcal{O}_X|D_+(f)$.*

Corollary (2.5.8). — *The restriction of $S(nd)\tilde{}$ to the open set $U = \bigcup_{f \in S_d} D_+(f)$ is invertible [i.e., locally free of rank 1 (0, 5.4.1)].*

Corollary (2.5.9). — *If S_1 generates S_+, then $S(n)\tilde{}$ is an invertible sheaf on $X = \mathrm{Proj}(S)$ for every n.*

(2.5.10). From now on we use the notation

$$\mathcal{O}_X(n) = S(n)\tilde{} \qquad (2.5.10.1)$$

and also, for any open $U \subseteq X$ and sheaf of $\mathcal{O}_X|U$ modules $\mathcal{F}$,

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X|U} (\mathcal{O}(n)|U). \qquad (2.5.10.2)$$

If S_1 generates S_+ then the functor $\mathcal{F} \mapsto \mathcal{F}(n)$ is exact.

(2.5.11). Given graded modules M, N, there are canonical functorial homomorphisms

$$\lambda_{(f)} \colon M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}, \qquad (2.5.11.1)$$

and hence

$$\lambda \colon \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to (M \otimes_S N)\tilde{}. \qquad (2.5.11.2)$$

If $\mathcal{I}$ and $\mathcal{J}$ are graded ideals, then since $\widetilde{\mathcal{I}}$, $\widetilde{\mathcal{J}}$ are ideal sheaves, there is a canonical homomorphism $\widetilde{\mathcal{I}} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{J}} \to \mathcal{O}_X$. It is equal to the composite

$$\widetilde{\mathcal{I}} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{J}} \xrightarrow{\lambda} (\mathcal{I} \otimes_S \mathcal{J})\tilde{} \to \mathcal{O}_X. \qquad (2.5.11.3)$$

Finally, given three graded modules, there is a canonical homomorphism

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \otimes_{\mathcal{O}_X} \widetilde{P} \to (M \otimes_S N \otimes_S P)^{\sim} \tag{2.5.11.4}$$

given by $\lambda \circ (\lambda \otimes 1) = \lambda \circ (1 \otimes \lambda)$.

(2.5.12). Similarly, there is a canonical functorial homomorphism of $S_{(f)}$ modules

$$\mu_{(f)} \colon \operatorname{Hom}_S(M, N)_{(f)} \to \operatorname{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)}), \tag{2.5.12.1}$$

and hence, using (I, 1.3.8), a canonical homomorphism of $\mathcal{O}_X$ module sheaves

$$\mu \colon \operatorname{Hom}_S(M, N)^{\sim} \to \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}). \tag{2.5.12.2}$$

Proposition (2.5.13). — *Suppose S_1 generates S_+. Then λ in (2.5.11.2) is an isomorphism; and if M is finitely presented (2.1.1), then so is μ in (2.5.12.2). If $\mathcal{I}$ is a graded ideal, then $\widetilde{\mathcal{I}}\widetilde{M} = (\mathcal{I}M)^{\sim}$.*

Corollary (2.5.14). — *If S_1 generates S_+, then there are canonical isomorphisms*

$$\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n) \tag{2.5.14.1}$$

$$\mathcal{O}_X(n) \cong (\mathcal{O}_X(1))^{\otimes n} \tag{2.5.14.2}$$

for all integers m, n.

Corollary (2.5.15). — *If S_1 generates S_+, then there is a canonical isomorphism $M(n)^{\sim} \cong \widetilde{M}(n)$, for every graded module M.*

(2.5.16). Under the identifications $X = \operatorname{Proj}(S) \cong X' = \operatorname{Proj}(S') \cong X^{(d)} = \operatorname{Proj}(S^{(d)})$ of (2.4.7), we have $\mathcal{O}_X(n) \cong \mathcal{O}_{X'}(n)$ and $\mathcal{O}_{X^{(d)}}(n) \cong \mathcal{O}_X(nd)$.

Proposition (2.5.17). — *The canonical homomorphisms $\mathcal{O}_X(nd) \otimes_{\mathcal{O}} \mathcal{O}_X(md) \to \mathcal{O}_X((m+n)d)$ restrict to isomorphisms on $U = \bigcup_{f \in S_d} D_+(f)$.*

2.6. **Graded S module associated to a sheaf on** $\operatorname{Proj}(S)$**.**

In this section we assume that S_1 generates the ideal S_+, and put $X = \operatorname{Proj}(S)$.

(2.6.1). By (2.5.9), the sheaf $\mathcal{O}_X(1)$ is invertible. For any $\mathcal{O}_X$ module sheaf $\mathcal{F}$ we define as in (0, 5.4.6)

$$\Gamma_*(\mathcal{F}) = \Gamma_*(\mathcal{O}_X(1), \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)), \tag{2.6.1.1}$$

the second equality following from (2.5.14.2). Then (0, 5.4.6) $\Gamma_*(\mathcal{O}_X)$ is a graded ring and $\Gamma_*(\mathcal{F})$ is a graded $\Gamma_*(\mathcal{O}_X)$ module sheaf. Since $\mathcal{O}_X(n)$ is locally free, $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$ is a left exact functor. In particular, if $\mathcal{I}$ is an ideal sheaf, then $\Gamma_*(\mathcal{I})$ is a graded ideal in $\Gamma_*(\mathcal{O}_X)$.

(2.6.2). The map $x \mapsto x/1\colon M_0 \to M_{(f)}$ induces maps $M_0 \to \Gamma(D_+(f), \widetilde{M})$ for all homogeneous $f \in S_+$, compatible with restricions, and hence a map

$$\alpha_0\colon M_0 \to \Gamma(X, \widetilde{M}). \tag{2.6.2.1}$$

Applying this to $M(n)$ and using (2.5.15), we get

$$\alpha_n\colon M_n = M(n)_0 \to \Gamma(X, \widetilde{M}(n)), \tag{2.6.2.2}$$

and hence a homomorphism of graded abelian groups

$$\alpha\colon M \to \Gamma_*(\widetilde{M}). \tag{2.6.2.3}$$

The map $\alpha\colon S \to \Gamma_*(\mathcal{O}_X)$ is a graded ring homomorphism, and (2.6.2.3) is an S module homomorphism.

Proposition (2.6.3). — *For every $f \in S_d$ ($d > 0$), the open set $D_+(f)$ is the non-vanishing locus of the section $\alpha(f)$ of $\mathcal{O}_X(d)$ (0, 5.5.2).*

(2.6.4). Set $M = \Gamma_*(\mathcal{F})$, which we may consider as an S module via $S \to \Gamma_*(\mathcal{O}_X)$. By (2.6.3), the section $\alpha_d(f)$ of $\mathcal{O}_X(d)$ is invertible on $D_+(f)$. Hence there is an $S_{(f)}$ module homomorphism

$$\beta_{(f)}\colon M_{(f)} \to \Gamma(D_+(f), \mathcal{F}) \tag{2.6.4.1}$$

given by $z/f^n \mapsto (z|D_+(f))/(\alpha_d(f)|D_+(f))^n$. This is compatible with restriction to $D_+(fg)$, giving a canonical homomorphism of sheaves of $\mathcal{O}_X$ modules

$$\beta\colon \Gamma_*(\mathcal{F})^{\sim} \to \mathcal{F}. \tag{2.6.4.2}$$

Proposition (2.6.5). — *For any graded S module M and sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$, each of the following maps is the identity:*

$$\widetilde{M} \xrightarrow{\tilde{\alpha}} \Gamma_*(\widetilde{M})^{\sim} \xrightarrow{\beta} \widetilde{M},$$

$$\Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*(\Gamma_*(\mathcal{F})^{\sim}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F})$$

2.7. **Finiteness conditions.**

Proposition (2.7.1). — *(i) If S is a Noetherian graded ring, then $X = \mathrm{Proj}(S)$ is a Noetherian scheme.*

(ii) If S is a finitely-generated graded A-algebra, then X is a scheme of finite type over $Y = \mathrm{Spec}(A)$.

(2.7.2). Consider two conditions on a graded S module M:

(TF) There exists n such that $\bigoplus_{k \geq n} M_k$ is a finitely generated S module;

(TN) There exists n such that $M_k = 0$ for $k \geq n$.

A graded S module homomorphism u will be called (TN)-*injective* (resp. (TN)-*surjective*, (TN)-*bijective*) if its kernel (resp. cokernel, both) satisfies (TN). By (2.5.4), this implies that $\widetilde{u}$ is injective (resp. surjective, bijective).

Proposition (2.7.3). — *Assume that S_+ is a finitely generated ideal.*

(i) If M satisfies (TF), then $\widetilde{M}$ is an $\mathcal{O}_X$ module of finite type.

(ii) If M satisfies (TF), then $\widetilde{M} = 0$ if and only if M satisfies (TN).

Corollary (2.7.4). — *If S_+ is finitely generated, then* $\operatorname{Proj}(S) = \emptyset$ *iff there is an n such that $S_k = 0$ for all $k \geq n$.*

Theorem (2.7.5). — *Let $X = \operatorname{Proj}(S)$, where S_+ is generated by finitely many elements, homogeneous of degree* 1*. Then for every quasi-coherent sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$, the canonical homomorphism $\beta\colon \Gamma_*(\mathcal{F})^{\sim} \to \mathcal{F}$ (2.6.4) is an isomorphism.*

Remark (2.7.6). — If S is Noetherian and S_1 generates S_+, then the hypotheses of (2.7.5) hold.

Corollary (2.7.7). — *Under the hypotheses of (2.7.5), every quasi-coherent $\mathcal{O}_X$ module $\mathcal{F}$ is isomorphic to $\widetilde{M}$ for some graded S module M.*

Corollary (2.7.8). — *Under the hypotheses of (2.7.5), every quasi-coherent $\mathcal{O}_X$ module $\mathcal{F}$ of finite type is isomorphic to $\widetilde{N}$ for some finitely generated graded S module N.*

Corollary (2.7.9). — *Under the hypotheses of (2.7.5), let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ module of finite type. Then there exists n_0 such that for all $n \geq n_0$, $\mathcal{F}(n)$ is isomorphic to a quotient of $\mathcal{O}_X^k$ (where k depends on n), i.e., $\mathcal{F}(n)$ is generated by finitely many global sections (0, 5.1.1).*

Corollary (2.7.10). — *Under the hypotheses of (2.7.5), let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ module of finite type. Then there exists n_0 such that for all $n \geq n_0$, $\mathcal{F}$ is isomorphic to a quotient of $\mathcal{O}_X(-n)^k$ (where k depends on n).*

Proposition (2.7.11). — *Assume the hypotheses of (2.7.5) hold, and let M be a graded S module.*

(i) The canonical homomorphism $\widetilde{\alpha}\colon \widetilde{M} \to \Gamma_*(\widetilde{M})^{\sim}$ is an isomorphism.

(ii) Let $\mathcal{G} \subseteq \widetilde{M}$ be a quasi-coherent $\mathcal{O}_X$ submodule sheaf, and let $N \subseteq M$ be the preimage of $\Gamma_*(\mathcal{G}) \subseteq \Gamma_*(\widetilde{M})$ via α. Then $\widetilde{N} = \mathcal{G}$.

2.8. **Functorial behavior.**

(2.8.1). Let $\phi\colon S' \to S$ be a graded ring homomorphism. Let $G(\phi)$ denote the complement of $V_+(\phi(S'_+))$ in $X = \operatorname{Proj}(S)$, that is, the union of the open sets $D_+(\phi(f'))$ for homogeneous $f' \in S_+$. Then ${}^a\phi\colon \operatorname{Spec}(S) \to \operatorname{Spec}(S')$ induces a continuous map ${}^a\phi\colon G(\phi) \to \operatorname{Proj}(S')$ such that

$$ {}^a\phi^{-1}(D_+(f')) = D_+(\phi(f')). \qquad (2.8.1.1)$$

Let $f = \phi(f')$. Then ϕ induces $\phi_f\colon S'_{f'} \to S_{\phi(f')}$ and $\phi_{(f)}\colon S'_{(f')} \to S_{(\phi(f'))}$, hence a morphism ${}^a\phi_{(f)}\colon D_+(f) \to D_+(f')$, which on the underlying space is the restriction of ${}^a\phi$ to the open sets in (2.8.1.1). These are compatible with restriction to $D_+(fg)$.

Proposition (2.8.2). — *There is a unique morphism* $({}^a\phi, \widetilde{\phi})\colon G(\phi) \to \mathrm{Proj}(S')$ *(called the morphism associated to* ϕ *and denoted* $\mathrm{Proj}(\phi)$*) whose restriction to each* $D_+(\phi(f'))$ *coincides with* ${}^a\phi_{(f)}$.

Corollary (2.8.3). — *(i)* $\mathrm{Proj}(\phi)$ *is an affine morphism.*

(ii) If $\ker(\phi)$ *is nilpotent (in particular, if* ϕ *is injective), then* $\mathrm{Proj}(\phi)$ *is dominant.*

In general a morphism $\mathrm{Proj}(S) \to \mathrm{Proj}(S')$ need not be affine, hence not of the form $\mathrm{Proj}(\phi)$. An example is $\mathrm{Proj}(S) \to \mathrm{Spec}(A) = \mathrm{Proj}(A[t])$ when S is an A-algebra.

(2.8.4). Given a third ring S'' and $\phi'\colon S'' \to S'$, let $\phi'' = \phi \circ \phi'$. Then $G(\phi'') \subseteq G(\phi)$, and if Φ, Φ', Φ'' are the associated morphisms, then $\Phi'' = \Phi' \circ (\Phi|G(\phi''))$.

(2.8.5). Suppose S (resp. S') is a graded A-algebra (resp. A'-algebra), and $\psi\colon A' \to A$ commutes with $\phi\colon S' \to S$. Then $G(\phi)$ and $\mathrm{Proj}(S')$ are schemes over $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A')$ respectively, and the the corresponding diagram commutes.

(2.8.6). Let M be a graded S module, which we may consider as a graded S' module $M_{[\phi]}$.

Proposition (2.8.7). — *There is a canonical functorial isomorphism* $(M_{[\phi]})^{\sim} \cong \Phi_*(\widetilde{M}|G(\phi))$, *where* $\Phi = \mathrm{Proj}(\phi)$.

Proposition (2.8.8). — *Let* M' *be a graded* S' *module. There is a canonical functorial homomorphism* $\nu\colon \Phi^*(\widetilde{M'}) \to (M' \otimes_{S'} S)^{\sim}|G(\phi)$. *If* S'_1 *generates* S'_+, *then* ν *is an isomorphism.*

(2.8.9). Let $\psi\colon A' \to A$ be a ring homomorphism, $\Psi\colon Y = \mathrm{Spec}(A) \to \mathrm{Spec}(A') = Y'$ its associated morphism. Let S' be a positively graded A'-algebra; then $S = S' \otimes_{A'} A$ is a positively graded A-algebra. We have the ring homomorphism $\phi\colon S' \to S$, $\phi(s') = s' \otimes 1$, and $\phi(S'_+)$ generates S_+ as an A module, hence $G(\phi) = \mathrm{Proj}(S) = X$. Set $X' = \mathrm{Proj}(S')$. Further, let M' be a graded S' module, and set $M = M' \otimes_{A'} A = M' \otimes_{S'} S$.

Proposition (2.8.10). — *With the notation of (2.8.9), we have* $X = X' \times_{Y'} Y$, *and the canonical homomorphism* $\nu\colon \Phi^*(\widetilde{M'}) \to \widetilde{M}$ *(2.8.8) is an isomorphism.*

Corollary (2.8.11). — *For all* $n \in \mathbb{Z}$, $\widetilde{M}(n)$ *is identified with* $\Phi^*(\widetilde{M'}(n)) = \widetilde{M'}(n) \otimes_{Y'} \mathcal{O}_Y$. *In particular,* $\mathcal{O}_X(n) = \Phi^*\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(n) \otimes_{Y'} \mathcal{O}_Y$.

(2.8.12). For $f' \in S'_d$ $(d > 0)$ and $f = \phi(f')$, the canonical map $M'_{(f')} \to M_{(f)}$ is identified with $M'^{(d)}/(f'-1)M'^{(d)} \to M^{(d)}/(f-1)M^{(d)}$ by (2.2.5).

(2.8.13). In the setting of (2.8.9), let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$ module, and set $\mathcal{F} = \Phi^*(\mathcal{F}')$. Then $\mathcal{F}(n) = \Phi^*(\mathcal{F}'(n))$ by (2.8.11) and (0, 4.3.3). From (0, 4.4.3) we have $\Gamma(\rho)\colon \Gamma(X', \mathcal{F}'(n)) \to \Gamma(X, \mathcal{F}(n))$ for all $n \in \mathbb{Z}$, giving a homomorphism of graded modules $\Gamma_*(\mathcal{F}') \to \Gamma_*(\mathcal{F})$.

If S_1 generates S_+ and $\mathcal{F}' = \widetilde{M}'$, then $\mathcal{F} = \widetilde{M}$, where $M = M' \otimes_{A'} A$, and we have commutative diagrams

(2.8.13.1)
$$\begin{array}{ccc} M' & \xrightarrow{\alpha_{M'}} & \Gamma_*(\widetilde{M}') \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha_M} & \Gamma_*(\widetilde{M}), \end{array}$$

(2.8.13.2)
$$\begin{array}{ccc} \Gamma_*(\mathcal{F}')^\sim & \xrightarrow{\beta_{\mathcal{F}'}} & \mathcal{F}' \\ \downarrow & & \downarrow \\ \Gamma_*(\mathcal{F})^\sim & \xrightarrow{\beta_{\mathcal{F}}} & \mathcal{F}, \end{array}$$

in which the vertical arrows are Φ-morphisms.

(2.8.14). Given a second graded S' module N', we have a canonical homomorphism

(2.8.14.1)
$$\Phi^*((M' \otimes_{S'} N')^\sim) \to (M \otimes_S N)^\sim,$$

and a commutative diagram

(2.8.14.2)
$$\begin{array}{ccc} \Phi^*(\widetilde{M}' \otimes_{\mathcal{O}_{X'}} \widetilde{N}') & \xrightarrow{\sim} & \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \\ {\scriptstyle \Phi^*(\lambda)}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \Phi^*((M' \otimes_{S'} N')^\sim) & \longrightarrow & (M \otimes_S N)^\sim, \end{array}$$

where the top row is the canonical isomorphism (0, 4.3.3). If S'_1 generates S'_+, then S_1 generates S_+, the vertical arrows are isomorphisms by (2.5.13), and hence (2.8.14.1) is an isomorphism.

Similarly, there is a commutative diagram

$$\begin{array}{ccc} \Phi^*(\mathrm{Hom}_{S'}(M', N')^\sim) & \longrightarrow & \mathrm{Hom}_S(M, N)^\sim \\ {\scriptstyle \Phi^*(\mu)}\downarrow & & \downarrow{\scriptstyle \mu} \\ \Phi^*(\mathcal{H}om_{\mathcal{O}_{X'}}(\widetilde{M}', \widetilde{N}')) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}), \end{array}$$

with bottom row given by (0, 4.4.6) and vertical arrows by (2.5.12).

(2.8.15). One can replace S_0 and S'_0 by $\mathbb{Z}$, or replace S and S' by $S^{(d)}$ and $S'^{(d)}$ as in (2.4.7), without changing Φ.

2.9. **Closed subschemes of** Proj(S)**.**

(2.9.1). If $\phi\colon S' \to S$ is (TN)-*injective* (resp. (TN)-surjective, (TN)-bijective) (2.7.2), then (2.8.15) shows that where Φ is concerned we can reduce to the case that ϕ is actually injective (resp. surjective, bijective).

Proposition (2.9.2). — *Let $X = \mathrm{Proj}(S)$.*

(i) If $\phi\colon S \to S'$ is (TN)-surjective, then the associated morphism Φ is defined on all of $\mathrm{Proj}(S')$ and is a closed immersion into X. If $\mathcal{I} = \ker(\phi)$, the image of Φ is the closed subscheme defined by the ideal sheaf $\widetilde{\mathcal{I}}$.

(ii) Suppose further that S_+ is generated by finitely many elements, homogeneous of degree 1. Let $X' \subseteq X$ be a closed subscheme, defined by a quasi-coherent sheaf of ideals $\mathcal{J}$, and let $\mathcal{I} \subseteq S$ be the preimage of $\Gamma_*(\mathcal{J})$ under $\alpha\colon S \to \Gamma_*(\mathcal{O}_X)$ (2.6.2). Set $S' = S/\mathcal{I}$. Then X' is the image of the closed immersion $\mathrm{Proj}(S') \to X$ associated to the canonical surjection $S \to S'$.

Corollary (2.9.3). — *In (2.9.2 (i)), if S_1 generates S_+, then $\Phi^*(S(n)^{\sim}) = S'(n)^{\sim}$ for all n, and $\Phi^*(\mathcal{F}(n)) = (\Phi^*(\mathcal{F}))(n)$ for every $\mathcal{O}_X$ module sheaf $\mathcal{F}$.*

Corollary (2.9.4). — *In (2.9.2 (ii)), the subscheme X' is integral if and only if the ideal $\mathcal{I}$ is prime.*

["If" is clear from (2.4.4). "Only if" uses (I, 7.4.4).]

Corollary (2.9.5). — *Let S be a graded A-algebra which is generated by S_1, M an A module, and $u\colon M \to S_1$ a surjective A module homomorphism, inducing $\overline{u}\colon \mathbf{S}(M) \to S$, where $\mathbf{S}(M)$ is the symmetric algebra of M. Then $\overline{u}$ induces a closed immersion of $\mathrm{Proj}(S)$ into $\mathrm{Proj}(\mathbf{S}(M))$.*