Synopsis of material from EGA Chapter II, §2.1–2.4

2. Homogeneous prime spectra

2.1. **Generalities on graded rings and modules.**

(2.1.1). *Notation.* Let S be an non-negatively graded ring. Its degree n component is denoted S_n. The subset $S_+ = \bigoplus_{n>0} S_n$ is a graded ideal and S_0 is a subring. The degree n component M_n of a graded S module M is an S_0 submodule, for every $n \in \mathbb{Z}$. By convention we set $S_n = 0$ for $n < 0$ when considering S as a graded S module.

For every $d > 0$, we have a graded subring $S^{(d)} = \bigoplus_n S_{nd}$ of S, and for each integer $0 \le k < d$, a graded $S^{(d)}$ submodule $M^{(d,k)} = \bigoplus_n M_{nd+k}$ of M. We write $M^{(d)}$ for $M^{(d,0)}$.

The degree shifted module $M(n)$ is defined by $(M(n))_k = M_{n+k}$. A graded module M isomorphic to a direct sum $\bigoplus_i S(n_i)$ is said to be *free*. This is equivalent to M admitting a homogeneous basis.

A graded module M is *finitely presented* if there is an exact sequence $P \to Q \to M$, where P, Q are finitely-generated free graded modules, and the maps are homogeneous of degree zero.

(2.1.2). $M \otimes_S N$ is graded, with $(M \otimes_S N)_n$ the image of $\bigoplus_{p+q=n} M_p \otimes_{S_0} N_q$. Let H_n be the set of S-module homomorphisms $M \to N$ homogeneous of degree n. We set $\operatorname{Hom}_S(M,N) = \bigoplus_n H_n$; it is a graded S-module. In general, $\operatorname{Hom}_S(M,N)$ is a proper subset of the set of all S-module homomorphisms $M \to N$. They are equal if M is finitely generated. A *homomorphism of graded S-modules* from M to N is an element of $(\operatorname{Hom}_S(M,N))_0$. We have

$$M(m) \otimes_S N(n) = (M \otimes_S N)(m+n), \tag{2.1.2.1}$$

$$\operatorname{Hom}_S(M(m), N(n)) = (\operatorname{Hom}_S(M,N))(n-m) \tag{2.1.2.2}$$

If $\phi\colon S \to S'$ is a ring homomorphism homogeneous of degree 0, it makes S' a *graded S-algebra*. Then $M \otimes_S S'$ is naturally a graded S'-module.

Lemma (2.1.3). — *A set of homogeneous elements $E \subseteq S_+$ generates S_+ as an ideal if and only if it generates S as an S_0-algebra.*

Corollary (2.1.4). — *S_+ is a finitely generated ideal iff S is a finitely generated S_0-algebra.*

Corollary (2.1.5). — *S is Noetherian iff S_0 is Noetherian and S is a finitely generated S_0-algebra.*

Lemma (2.1.6). — *Let S be non-negatively graded and finitely generated as an S_0-algebra. Let M be a finitely generated graded S-module. Then:*

(i) Each M_n is a finitely generated S_0-module, and there exists an n_0 such that $M_n = 0$ for all $n < n_0$.

(ii) There exist $h > 0$ and n_1 such that $M_{n+h} = S_h M_n$ for all $n \ge n_1$.

(iii) For every $d > 0$ and $0 \le k < d$, $M^{(d,k)}$ is a finitely generated $S^{(d)}$-module.

(iv) For every $d > 0$, $S^{(d)}$ is a finitely generated S_0-algebra.

(v) There exists $h > 0$ such that $S_{mh} = (S_m)^h$ for all $m > 0$.
(vi) For every $n > 0$, there exists m_0 such that $S_m \subseteq (S_+)^n$ for all $m \geq m_0$.

Corollary (2.1.7). — *If S is Noetherian, so is every $S^{(d)}$.*

(2.1.8). Let $\mathfrak{p} = \oplus_n(\mathfrak{p} \cap S_n)$ be a graded prime ideal of S. Suppose $\mathfrak{p} \not\supseteq S_+$. If $f \in S_+ \setminus \mathfrak{p}$, we have $f^n x \in \mathfrak{p}$ iff $x \in \mathfrak{p}$. In particular, if $f \in S_d$ $(d > 0)$, and $x \in S_{m-nd}$, then $f^n x \in \mathfrak{p}_m$ iff $x \in \mathfrak{p}_{m-nd}$.

Proposition (2.1.9). — *Let $n_0 > 0$, and for all $n \geq n_0$ let $\mathfrak{p}_n$ be a subgroup of S_n. For there to exist a graded prime ideal $\mathfrak{p} \not\supseteq S_+$ such that $\mathfrak{p} \cap S_n = \mathfrak{p}_n$ for all $n \geq n_0$, it is necessary and sufficient that the following condintions hold.*
(i) $S_m \mathfrak{p}_n \subseteq \mathfrak{p}_{m+n}$ for all $n \geq n_0$ and $m \geq 0$.
(ii) For all $m, n \geq n_0$, $f \in S_m$, $g \in S_n$, $fg \in \mathfrak{p}_{m+n}$ implies $f \in \mathfrak{p}_m$ or $g \in \mathfrak{p}_n$.
(iii) $\mathfrak{p}_n \neq S_n$ for some $n \geq n_0$.
Moreover, $\mathfrak{p}$ is then unique.

(2.1.10). We call an ideal $\mathcal{I}$ of S contained in S_+ an *ideal of* S_+. If $\mathcal{I} = \mathfrak{p} \cap S_+$ for a graded prime ideal $\mathfrak{p} \not\supseteq S_+$ (unique by (2.1.9)), we call $\mathcal{I}$ a *graded prime ideal of* S_+.

We define the *radical in* S_+ of an ideal $\mathcal{I}$ of S_+ to be $\sqrt{}_+\mathcal{I} = (\sqrt{\mathcal{I}}) \cap S_+$. If $\mathcal{I}$ is graded, then so is $\sqrt{}_+\mathcal{I}$. The *nilradical of* S_+ is $\mathfrak{N}_+ = \sqrt{}_+ 0$. We say that S is *essentially reduced* if $\mathfrak{N}_+ = 0$.

(2.1.11). We call S *essentially integral* if S_+, considered as a ring without unit, is not zero and has no zero-divisors. Since the highest degree component of a zero-divisor $x \in S$ is again a zero-divisor, it suffices that $S_+ \neq 0$ and S has no homogeneous zero-divisors of degree > 0. If $\mathfrak{p}$ is a graded prime ideal of S_+, then $S/\mathfrak{p}$ is essentially integral.

Suppose S essentially integral. If $x_0 \in S_0$, and $x_0 f = 0$ for some homogeneous $f \in S_+$, it follows that $x_0 S_+ = 0$. Hence S is an integral domain if and only S_0 is an integral domain and the annihilator of S_+ is zero.

2.2. **Rings of fractions of a graded ring.**

(2.2.1). If S is graded, $f \in S_d$, $d > 0$, then S_f is $\mathbb{Z}$-graded, with $(S_f)_n = \{x/f^k \colon x \in S_{n+kd}\}$. We put $S_{(f)} = (S_f)_0$. The monomials $(f/1)^h$ $(h \in \mathbb{Z})$ are a basis of $(S^{(d)})_f$ as a free module over $S_{(f)}$; thus $(S^{(d)})_f \cong S_{(f)}[t, t^{-1}]$.

If M is a graded S-module, then M_f is a graded S_f-module, with $(M_f)_n = \{z/f^k : z \in M_{n+kd}\}$. We denote by $M_{(f)}$ the $S_{(f)}$ module $(M_f)_0$. Then $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

Lemma (2.2.2). — *Let $f \in S_d$, $g \in S_e$, $d, e > 0$. There is a canonical isomorphism of rings $S_{(fg)} \cong (S_{(f)})_{g^d/f^e}$, and after identifying these rings, a canonical isomorphism of modules $M_{(fg)} \cong (M_{(f)})_{g^d/f^e}$.*

(2.2.3). The canonical homomorphism $S_{(f)} \to (S_{(f)})_{g^d/f^e} \cong S_{(fg)}$ is the degree zero part of $S_f \to S_{fg}$, mapping x/f^k to $g^k x/(fg)^k$. Likewise for modules.

Lemma (2.2.4). — *For $f, g \in S_+$ homogeneous, the ring $S_{(fg)}$ is generated by the images of the canonical homomorphisms from $S_{(f)}$ and $S_{(g)}$.*

Proposition (2.2.5). — *Let $f \in S_d$, $d > 0$. There is a canonical isomorphism of rings $S_{(f)} \cong S^{(d)}/(f-1)S^{(d)}$, and after identifying them, a canonical isomorphism of modules $M_{(f)} \cong M^{(d)}/(f-1)M^{(d)}$.*

Corollary (2.2.6). — *If S is Noetherian, so is $S_{(f)}$ for all f homogeneous of degree > 0.*

(2.2.7). If T is a multiplicative set of homogeneous elements in S_+, then $T_0 = T \cup \{1\}$ is a multiplicative set in S; $T_0^{-1}S$ is graded in the obvious way; we set $S_{(T)} = (T_0^{-1}S)_0$. Since $T_0^{-1}S$ is the inductive limit of the rings S_f, as f ranges over T_0, therefore $S_{(T)}$ is the inductive limit of the rings $S_{(f)}$. Defining $M_{(T)}$ similarly, it is the inductive limit of the modules $M_{(f)}$. If $\mathfrak{p}$ is a graded prime ideal of S_+, then $S_{(\mathfrak{p})}$, $M_{(\mathfrak{p})}$ stand for $S_{(T)}$, $M_{(T)}$, where T is the set of homogeneous elements in $S_+ \setminus \mathfrak{p}$.

2.3. **Homogeneous prime spectrum of a graded ring.** [see also Liu, §2.3.3]

(2.3.1). Given a non-negatively graded ring S, its *homogeneous prime spectrum* $\mathrm{Proj}(S)$ is the set of graded prime ideals of S_+ (2.1.10), that is, the set of graded prime ideals of S not containing S_+. We will make $\mathrm{Proj}(S)$ the underlying set of a scheme.

(2.3.2). For any subset $E \subseteq S$, define $V_+(E)$ to be the set of graded primes of S such that $E \subseteq \mathfrak{p} \not\supseteq S_+$, that is, $V_+(E) = V(E) \cap \mathrm{Proj}(S) \subseteq \mathrm{Spec}(S)$. Using (I, 1.1.2), we see that the sets $V_+(E)$ are the closed sets of the subspace topology on $\mathrm{Proj}(S) \subseteq \mathrm{Spec}(S)$. Let $\mathcal{I}$ be the graded ideal generated by all homogeneous components of elements of E. Then $V_+(\mathcal{I}) = V_+(E)$. For any graded ideal $\mathcal{I} \subseteq S$ and integer n, we have $V_+(\mathcal{I}) = V_+(\bigcup_{q \geq n}(\mathcal{I} \cap S_q))$. Finally $V_+(\mathcal{I}) = V_+(\sqrt{_+\mathcal{I}})$.

(2.3.3). We regard $\mathrm{Proj}(S)$ as a topological space with closed subsets $V_+(E)$. For $f \in S$, define

$$D_+(f) = D(f) \cap \mathrm{Proj}(S) = \mathrm{Proj}(S) - V_+(f). \tag{2.3.3.1}$$

Then

$$D_+(fg) = D_+(f) \cap D_+(g) \tag{2.3.3.2}$$

Proposition (2.3.4). — *As f ranges over homogeneous elements of S_+, the sets $D_+(f)$ form a base for the topology on* $\mathrm{Proj}(S)$.

(2.3.5). Let $f \in S_d$ $(d > 0)$, and let $\mathfrak{p}$ be a graded prime ideal of S such that $f \notin \mathfrak{p}$. Then $\mathfrak{p}_f$ is a prime ideal in S_f (0, 1.2.6), and $\psi_f(\mathfrak{p}) = \mathfrak{p}_f \cap S_{(f)}$ is a prime ideal in $S_{(f)}$, consisting of elements x/f^n where $x \in \mathfrak{p} \cap S_{nd}$. This defines a map

$$\psi_f \colon D_+(f) \to \mathrm{Spec}(S_{(f)}).$$

Given $g \in S_e$ $(e > 0)$, we get a commutative diagram

$$\begin{array}{ccc} D_+(f) & \xrightarrow{\psi_f} & \mathrm{Spec}(S_{(f)}) \\ \uparrow & & \uparrow \\ D_+(fg) & \xrightarrow{\psi_{fg}} & \mathrm{Spec}(S_{(fg)}). \end{array} \tag{2.3.5.1}$$

Proposition (2.3.6). — *[Liu, 2.3.36(a)]* $\psi_f\colon D_+(f) \to \mathrm{Spec}(S_{(f)})$ *is a homeomorphism.*

Corollary (2.3.7). — $D_+(f) = \emptyset$ *if and only if* f *is nilpotent.*

Corollary (2.3.8). — *Let* $E \subseteq S_+$*. The following are equivalent:*
(a) $V_+(E) = X = \mathrm{Proj}(S)$*.*
(b) Every element of E *is nilpotent.*
(c) The homogeneous components of every element of E *are nilpotent.*

Corollary (2.3.9). — *If* $\mathcal{I}$ *is a graded ideal of* S_+*, then* $\sqrt{_+\mathcal{I}}$ *is the intersection of the the graded prime ideals of* S_+ *which contain* $\mathcal{I}$*.*

(2.3.10). For any subset $Y \subseteq \mathrm{Proj}(S)$, let $\mathfrak{j}_+(Y) = \{f \in S_+ : Y \subseteq V_+(f)\} = \mathfrak{j}(Y) \cap S_+$, an ideal of S_+ equal to its radical in S_+.

Proposition (2.3.11). — *(i) For every* $E \subseteq S_+$*,* $\mathfrak{j}_+(V_+(E)) = \sqrt{_+\mathcal{I}}$*, where* $\mathcal{I}$ *is the ideal generated by the homogeneous components of elements of* E*.*
(ii) For every $Y \subseteq \mathrm{Proj}(S)$*,* $V_+(i_+(Y))$ *is the closure of* Y *in* $\mathrm{Proj}(S)$*.*

Corollary (2.3.12). — *There is a containment-reversing bijection* $Y \to \mathfrak{j}_+(Y)$*,* $\mathcal{I} \to V_+(\mathcal{I})$ *between radical ideals of* S_+ *and closed subsets of* $\mathrm{Proj}(S)$*. The union* $Y_1 \cup Y_2$ *corresponds to the intersection of ideals; the intersection of any family* (Y_λ) *corresponds to the radical of the sum of their ideals.*

Corollary (2.3.13). — *Let* $\mathcal{I}$ *be a graded ideal of* S_+*. Then* $V_+(\mathcal{I}) = \emptyset$ *iff every element of* S_+ *has a power in* $\mathcal{I}$*.*

Corollary (2.3.14). — *Let* (f_α) *be graded elements of* S_+*. The open sets* $D_+(f_\alpha)$ *cover* $\mathrm{Proj}(S)$ *iff every element of* S_+ *has a power in the ideal generated by the* f_α*'s.*

Corollary (2.3.15). — *With* (f_α) *as in (2.3.14), and* $f \in S_+$*, the following are equivalent: (a)* $D_+(f) \subseteq \bigcup_\alpha D_+(f_\alpha)$*; (b)* $V_+(f) \supseteq \bigcap_\alpha V_+(f_\alpha)$*; (c) some power of* f *is in the ideal generated by the* f_α*'s.*

Corollary (2.3.16). — $\mathrm{Proj}(S) = \emptyset$ *iff every element of* S_+ *is nilpotent.*

Corollary (2.3.17). — *In the correspondence of (2.3.12), the irreducible closed subsets correspond to the graded prime ideals of* S_+*.*

2.4. **The scheme structure on** $\mathrm{Proj}(S)$**.**

(2.4.1). Let $f, g \in S_+$ be homogeneous. We can carry the structure of affine scheme on $Y_f = \mathrm{Spec}(S_{(f)})$ to $D_+(f)$ via the homeomorphism ψ_f in (2.3.6). By (2.2.2) and (2.3.5), if we make $D_+(fg)$ an affine scheme in the same way, then its inclusion into $D_+(f)$ is an open immersion. In particular, the scheme structures on $D_+(f)$ and $D_+(g)$ agree on $D_+(fg)$. This makes $\mathrm{Proj}(S)$ a prescheme.

Proposition (2.4.2). — *The prescheme* $\mathrm{Proj}(S)$ *is a scheme [i.e., it is separated].*

This follows from (I, 5.5.6) and (2.2.4).

Example (2.4.3). — For $S = K[t_1, t_2]$, $\mathrm{Proj}(S)$ is the projective line from (I, 2.3.2).

Proposition (2.4.4). — *Let S be a non-negatively graded ring, $X = \mathrm{Proj}(S)$.*

(i) Let $\mathfrak{N}_+$ be the nilradical of S_+. Then $X_{\mathrm{red}} = \mathrm{Proj}(S/\mathfrak{N}_+)$.

(ii) Suppose S essentially reduced. Then X is integral iff S is essentially integral.

(2.4.5). A *graded A-algebra* is a graded ring S with an A-algebra structure such that each S_n is an A-submodule; equivalently the structural homomorphism $A \to S$ has image in S_0.

Proposition (2.4.6). — *If S is a graded A-algebra, $X = \mathrm{Proj}(S)$, then $\mathcal{O}_X$ is a sheaf of A-algebras; that is, X is a scheme over* $\mathrm{Spec}(A)$.

Proposition (2.4.7). — *Let S be a non-negatively graded ring.*

(i) For all $d > 0$, $\mathrm{Proj}(S)$ is canonically isomorphic to $\mathrm{Proj}(S^{(d)})$.

(ii) Let S' be the graded ring with $S'_0 = \mathbb{Z}$, $S'_n = S_n$ for all $n > 0$. Then $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$.

Corollary (2.4.8). — *If S is a graded A-algebra, and S'_A is the graded A-algebra with $(S'_A)_0 = A$, $S'_n = S_n$ for $n > 0$, then $\mathrm{Proj}(S) \cong \mathrm{Proj}(S'_A)$.*