Synopsis of material from EGA Chapter I, §3

3. Product of preschemes

3.1. **Disjoint union of preschemes.**

(3.1.1). The disjoint union of ringed spaces $X = \coprod_\alpha X_\alpha$ is defined in an obvious way, with each X_α open and closed in X. If each X_α is a prescheme, then so is X. To give a morphism $X \to Y$ it is equivalent to give morphisms $X_\alpha \to Y$ for each α. In particular, if each X_α is an S-prescheme, then so is X. If $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ are affine, then $X \coprod Y = \operatorname{Spec}(A \times B)$ is affine [this does not hold for infinite disjoint unions].

3.2. **Product of preschemes.**

Definition (3.2.1). — [Liu, 3.1.1] Given S-preschemes X, Y, we say that an S-prescheme Z, together with S-morphisms $p_1 \colon Z \to X$, $p_2 \colon Z \to Y$, is their *product*, if for every S-prescheme T, the correspondence $f \mapsto (p_1 \circ f, p_2 \circ f)$ is a bijection

$$\operatorname{Hom}_S(T, Z) \cong \operatorname{Hom}_S(T, X) \times \operatorname{Hom}_S(T, Y).$$

In other words, Z is the product of X and Y in the category of S-preschemes. In particular, the product is unique up to a canonical isomorphism. We usually denote it by $X \times_S Y$, suppressing the morphisms p_1, p_2 (called the *canonical projections*) from the notation. Given S-morphisms $g \colon T \to X$ and $h \colon T \to Y$, we write (g, h) for the corresponding morphism $f \colon T \to X \times_S Y$. Given $u \colon X \to X'$ and $v \colon Y \to Y'$, we write $u \times v$ for the induced morphism $X \times_S Y \to X' \times_S Y'$.

When $S = \operatorname{Spec}(A)$, we often write A in place of S in the notation.

Proposition (3.2.2). — $\operatorname{Spec}(B) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(C) = \operatorname{Spec}(B \otimes_A C)$.

This follows from (2.2.4) and the universal property of the tensor product [Liu, 2.3.23 and 1.1.14].

Corollary (3.2.3). — *In the setting of (3.2.2), if* $T = \operatorname{Spec}(D)$ *and* $g \colon T \to \operatorname{Spec}(B)$, $h \colon T \to \operatorname{Spec}(C)$ *correspond to A-algebra homomorphisms* $\rho \colon B \to D$, $\sigma \colon C \to D$, *then* $(g, h) \colon T \to \operatorname{Spec}(B) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(C)$ *corresponds to the A-algebra homormorphism* $\tau \colon B \otimes_A C \to D$ *such that* $\tau(b \otimes c) = \rho(b)\sigma(c)$.

Proposition (3.2.4). — *If* $f \colon S' \to S$ *is a monomorphism, and* X, Y *are S'-schemes, viewed as S-schemes via f, then a product* $X \times_S Y$ *in the category of S-schemes is also the product* $X \times_{S'} Y$ *in the category of S'-schemes, and conversely.*

[Recall that an arrow $f \colon S' \to S$ in any category is a *monomorphism* if the induced map $\operatorname{Hom}(T, S') \to \operatorname{Hom}(T, S)$ is injective for every object T.]

Corollary (3.2.5). — *If* $S' \subseteq S$ *is open, and* $f \colon X \to S$, $g \colon Y \to S$ *have images contained in* S', *then a product of X and Y as S-schemes is a product as S'-schemes, and conversely.*

Theorem (3.2.6). — *[Liu, 1.1.2] The product of S-schemes* $X \times_S Y$ *always exists.*

If all three schemes are affine, the result follows from (3.2.2). The strategy of the proof is to use gluing to reduce the general case to the affine case.

Corollary (3.2.7). — *If $Z = X \times_S Y$, $S' \subseteq S$ is open, and $U \subseteq X$, $V \subseteq Y$ are open subsets lying over S', then the open subscheme $p_1^{-1}(U) \cap p_2^{-1}(V)$ of Z is the product $U \times_{S'} V$. If $f\colon T \to X$, $g\colon T \to Y$ have images contained in U, V respectively, then (f, g) has image contained in $U \times_{S'} V$, and $(f, g)_{S'}$ is (f, g) considered as a map $T \to U \times_{S'} V$.*

(3.2.8). The product of two disjoint unions $X = \coprod X_\alpha$, $Y = \coprod_\beta Y_\beta$ of S-schemes is the disjoint union of the products $X_\alpha \times_S Y_\beta$.

3.3. **Formal properties of the product; change of base.**

(3.3.1). Everything in this section except (3.3.13) and (3.3.15) is valid in any category in which products of objects X, Y over an object S exist.

(3.3.2). $X \times_S Y$ is a covariant bifunctor of X and Y [Liu, 1.1.4 (c)].

Proposition (3.3.3). — *The projection p_1 (resp. p_2) is a functorial isomorphism of $X \times_S S$ (resp. $S \times_S X$) on X, with inverse $(1_X, \phi)$, where $\phi\colon X \to S$ is the structure morphism. In other words,*

$$X \times_S S = S \times_S X = X$$

up to canonical isomorphism.

Corollary (3.3.4). — *Under the identifications $X = X \times_S S$, $Y = S \times_S Y$, the projections $X \times_S Y \to X$, $X \times_S Y \to Y$ are identified with $(1_X, \phi_Y)$, $(\phi_X, 1_Y)$.*

(3.3.5). One can define the product of n S-preschemes in the obvious way; it exists and is associative, *e.g.*, $X_1 \times X_2 \times X_3 = (X_1 \times X_2) \times X_3 = X_1 \times (X_2 \times X_3)$, up to canonical natural isomorphisms.

(3.3.6–8). Suppose given $\phi\colon S' \to S$, making S' an S-prescheme. For any S-prescheme X, the product $X \times_S S'$ is an S'-prescheme with structure morphism the second projection. We denote $X \times_S S'$, regarded as a scheme over S', by $X_{(S')}$, or by $X_{(\phi)}$. One says that $X_{(S')}$ is obtained by *base change from S to S'* [Liu, 3.1.7].

Given an S-morphism $f\colon X \to Y$, we write $f_{(S')}\colon X_{(S')} \to Y_{(S')}$ for $f \times_S 1_{S'}$. Then base change is a covariant functor from S-preschemes to S'-preschemes.

Base change is right adjoint to the "forgetful" functor which regards every S'-prescheme as an S-prescheme via ϕ. That is, S'-morphisms $T \to X_{(S')}$ are in natural bijection with S-morphisms $T \to X$ when T is regarded as an S-scheme.

Proposition (3.3.9). — *('Transitivity of base change') Given $S'' \to S' \to S$, we have a canonical natural isomorphism $X_{(S'')} = (X_{(S')})_{(S'')}$.*

Corollary (3.3.10). — *There is a canonical natural isomorphism*

$$(X \times_S Y)_{(S')} = X_{(S')} \times_{S'} Y_{(S')}.$$

Corollary (3.3.11). — *If Y is an S-prescheme and $f\colon X \to Y$ is a morphism making X a Y-prescheme, and hence an S-prescheme, then there is a canonical natural isomorphism $X_{(S')} = X \times_Y Y_{(S')}$, and $f_{(S')}$ is identified with the projection $X \times_Y Y_{(S')} \to Y_{(S')}$.*

(3.3.12). If f and g are monomorphisms, so is $f \times_S g$. In particular, so is $f_{(S')}$ for any base change.

(3.3.13). If $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$ are affine, with $S' \to S$ given by a ring homomorphism $A \to A'$, making A' an A algebra, then given an S-prescheme X, we also write $X_{(A')}$ or $X \otimes_A A'$ for $X_{(S')}$, since when $X = \mathrm{Spec}(B)$, we have $X_{(S')} = \mathrm{Spec}(B \otimes_A A')$, the affine scheme associated to the A' algebra obtained from B by extension of scalars.

(3.3.14). With the notation of (3.3.6), for every S-morphism $f\colon S' \to X$, $f' = (f, 1_{S'})_S$ is an S'-morphism from S' to $X_{(S')}$, that is, a section of $X_{(S')}$ over S'. Conversely, given any such section, its composite with the projection $X_{(S')} \to X$ is an S-morphism $S' \to X$. Thus we have a canonical bijection

$$\mathrm{Hom}_S(S', X') \cong \mathrm{Hom}_{S'}(S', X_{(S')}).$$

The morphism f' is called the *graph morphism* of f and denoted Γ_f.

(3.3.15). As every prescheme X is uniquely a prescheme over $\mathbb{Z}$, (3.3.14) implies that the X-sections of $X \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ correspond bijectively to morphisms $X \to \mathrm{Spec}(\mathbb{Z}[t])$, thus to ring homomorphisms $\mathbb{Z}[t] \to \mathcal{O}_X(X)$, and thus to global sections of $\mathcal{O}_X$.

3.4. **Points of a prescheme with values in a prescheme; geometric points.**

(3.4.1). Denote by $X(T)$ the set $\mathrm{Hom}(T, X)$ of morphisms $T \to X$ of preschemes. Its elements are called *points of* X *with values in* T. For fixed X, $T \to X(T)$ is a contravariant functor in T, from preschemes to sets. A morphism $g\colon X \to Y$ induces a natural transformation of functors $X(T) \to Y(T)$.

(3.4.2). Given sets and maps $\phi\colon P \to R$, $\psi\colon Q \to R$, the subset $\{(p, q) \in P \times Q : \phi(p) = \psi(q)\}$ is called the *fiber product* of P and Q over R (relative to the given maps). We can interpret the definition (3.2.1) of the product of S-preschemes as the identity

$$(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T). \tag{3.4.2.1}$$

(3.4.3). If we fix S and consider only S-morphisms of S-preschemes, we write $X(T)_S$ for $\mathrm{Hom}_S(T, X)$, suppressing the S from the notation when it will not cause confusion. Its elements are the *points* (or *S-points*) of the S-prescheme X with value in the S-prescheme T. In particular, the S-sections of X are just the points of X with value in S. Then (3.4.2.1) may also be written

$$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S. \tag{3.4.3.1}$$

More generally if Z is an S-prescheme and X, Y, T are Z-preschemes (hence *ipso facto* S-preschemes), we have

$$(X \times_Z Y)(T)_S = X(T)_S \times_{Z(T)_S} Y(T)_S. \tag{3.4.3.2}$$

To show that an S-prescheme W with S-morphisms $p_1\colon W \to X$, $p_2\colon W \to Y$ is a product of X and Y over an S-prescheme Z, it suffices to show that for every S-prescheme T, the diagram

$$\begin{array}{ccc} W(T)_S & \longrightarrow & X(T)_S \\ \downarrow & & \downarrow \\ Y(T)_S & \longrightarrow & Z(T)_S \end{array}$$

is a fiber product of sets.

(3.4.4). In the preceding, when $T = \mathrm{Spec}(B)$ or $S = \mathrm{Spec}(A)$, we often replace T and/or S by B, A in the notation, and refer to the elements of $X(B)$, $X(B)_A$ as *points of* X *with values in* B, or *points of the* A*-prescheme* X *with values in the* A*-algebra* B. Note that $X(B)$, $X(B)_A$ are *covariant* functors from rings (resp. A-algebras) to sets.

(3.4.5). As a special case, if $T = \mathrm{Spec}(A)$ where A is a local ring, then the points in $X(A)$ correspond bijectively to pairs consisting of a point $x \in X$ and a ring homomorphism $\mathcal{O}_{X,x} \to A$; see (2.4.4). The point x is called the *location* of the corresponding point in $X(A)$.

Still more specially, points of X with values in a field K are called *geometric points*. A geometric point corresponds to a point $x \in X$ and a field extension $k(x) \hookrightarrow K$. One speaks of a geometric point *located at* x, with *value field* K. There is a map $X(K) \to X$ sending a geometric point to its location.

If $\mathrm{Spec}(K)$ is an S-prescheme, that is, K is an extension of $k(s)$ for some $s \in S$, and X is an S-prescheme, elements of $X(K)_S$ are *geometric points of* X *lying over* s *with values in* K. Such a point is given by its location $x \in X$, which must lie over $s \in S$, and a $k(s)$-homomorphism $k(x) \to K$; note that the structure morphism of X as an S-scheme makes $k(x)$ an extension of $k(s)$.

In particular, if $S = \mathrm{Spec}(K) = \{\xi\}$, then $X(K)_K$ is identified with the set of points of x such that $k(x) = K$, called K*-rational points of the* K*-prescheme* X. If K' is an extension of K, then $X(K')_K$ corresponds bijectively with the set of K'-rational points of $X_{(K')}$ (3.3.14).

Lemma (3.4.6). — *Let* $X_1, \ldots, X_n$ *be* S*-preschemes,* $s \in S$*,* $x_i \in X_i$ *lying over* s*. There exists an extension* K *of* $k(s)$ *and a geometric point of the product* $Y = X_1 \times_S \cdots \times_S X_n$ *with values in* K*, whose projection on each* X_i *is located at* x_i*.*

Proposition (3.4.7). — *Given* S*-preschemes* $X_1, \ldots, X_n$ *and points* $x_i \in X$*, there exists a point* y *of the product* $Y = X_1 \times_S \cdots \times_S X_n$ *whose projection on each* X_i *is* x_i*, if and only if all the points* x_i *lie over the same point* $s \in S$*.*

Denoting the underlying set of X by (X), the proposition says that $(X \times_S Y)$ maps *surjectively* on $(X) \times_{(S)} (Y)$. This map is *not injective* in general—more than one point of $X \times_S Y$ can have the same projections $x \in X$ and $y \in Y$. Example: let X, Y, S be the spectra of fields K, K', k. The algebra $K \otimes_k K'$ need not be a field, and can have more than one prime ideal—see (3.4.9).

Corollary (3.4.8). — *Let* $f_{(S')} \colon X_{(S')} \to Y_{(S')}$ *be obtained from a morphism* $f \colon X \to Y$ *of* S*-preschemes by base change. Let* $p \colon X_{(S')} \to X$*,* $q \colon Y_{(S')} \to Y$ *be the canonical projections. Then for every subset* $M \subseteq X$*, we have* $q^{-1}(f(M)) = f_{(S')}(p^{-1}(M))$*.*

Proposition (3.4.9). — *Let X, Y be S-preschemes, $x \in X$, $y \in Y$ lying over the same point $s \in S$. The points in $X \times_S Y$ which project on x and y correspond bijectively to isomorphism classes of extensions of $k(s)$ generated by $k(x)$ and $k(y)$, or equivalently, to the points of* $\operatorname{Spec}(k(x) \otimes_{k(s)} k(y))$.

3.5. **Surjections and injections.**

(3.5.1). Consider the following assertions about a property **P** of morphisms of preschemes:

(i) If f, g are S-morphisms with property **P**, then $f \times_S g$ has property **P**.

(ii) If f is an S-morphism with property **P**, then every base change $f_{(S')}$ has property **P**.

If the identity morphism 1_X on every scheme X has property **P**, then (i) implies (ii). If the composite of two morphisms with property **P** has property **P**, then (ii) implies (i). [When (ii) holds, **P** is called *stable under base change*—see Liu, 3.1.23.]

Proposition (3.5.2). — *(i) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are surjective S-morphisms, then $f \times_S g$ is surjective.*

(ii) If $f\colon X \to Y$ is a surjective S-morphism, then so is any base change $f_{(S')}$ [Liu, Ex. 3.1.8].

In fact, (3.4.8) with $M = X$ gives (ii), and then (3.5.1) gives (i).

Proposition (3.5.3). — *A morphism $f\colon X \to Y$ is surjective if and only if, for every field K and morphism* $\operatorname{Spec}(K) \to Y$, *there exists an extension K' of K and a morphism* $\operatorname{Spec}(K') \to X$ *making the diagram*

$$\begin{array}{ccc} \operatorname{Spec}(K') & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \operatorname{Spec}(K) & \longrightarrow & Y \end{array}$$

commute. In other words, f is surjective iff every geometric point of Y with values in K is the image of a geometric point of X with values in some extension of K.

Definition (3.5.4). — A morphism $f\colon X \to Y$ is *universally injective* (or *radicial*) if $X(K) \to Y(K)$ is injective for every field K.

In particular, a monomorphism of preschemes is universally injective.

(3.5.5). For f to be universally injective, it suffices that the condition hold for all algebraically closed K.

Proposition (3.5.6). — *(i) The composite of two universally injective morphisms is univerally injective.*

(ii) If $g \circ f$ is universally injective, then so is f.

Proposition (3.5.7). — *(i) If f, g are universally injective S-morphisms, then so is $f \times_S g$.*

(ii) If f is universally injective, then so is any base change $f_{(S')}$.

Proposition (3.5.8). — *$f\colon X \to Y$ is universally injective if and only if f is injective, and for every $x \in X$, the homomorphism $f^x\colon k(f(x)) \to k(x)$ makes $k(x)$ a purely inseparable algebraic extension of $k(f(x))$.*

[An algebraic extension of fields $K \subseteq L$ is *purely inseparable* if every element of $L \setminus K$ is inseparable over K. This is equivalent to L having a unique embedding (over K) into the algebraic closure of K. In characteristic zero, only trivial extensions $K = L$ are purely inseparable.]

Corollary (3.5.9). — *The canonical morphism* $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ *is universally injective.*

In fact, it is a monomorphism (1.6.2).

Corollary (3.5.10). — *Let $f\colon X \to Y$ be universally injective, $g\colon Y' \to Y$ any morphism, and put $X' = X_{(Y')} = X \times_Y Y'$. Then the universally injective morphism $f_{(Y')}$ is a bijection of the underlying set of X' onto $g^{-1}(f(X))$. Moreover, for any field K, $X'(K)$ is identified with the subset of $Y'(K)$ which is the preimage of $X(K) \subseteq Y(K)$ via the map $Y'(K) \to Y(K)$ induced by g.*

(3.5.11). A morphism $f = (\psi, \phi)\colon X \to Y$ is said to be *injective* if ψ is injective. Then f is universally injective if and only if every base change $f_{(Y')}\colon X_{(Y')} \to Y'$ is injective; this explains the terminology.

3.6. **Fibers.**

Proposition (3.6.1). — *[Liu, 3.1.16] Suppose given a morphism $f\colon X \to Y$, a point $y \in Y$, and an ideal $\mathfrak{a}_y$ in $\mathcal{O}_y$ which contains a power of the maximal ideal $\mathfrak{m}_y$. Then the projection $p\colon X \times_Y \mathrm{Spec}(\mathcal{O}_y/\mathfrak{a}) \to X$ is a homeomorphism onto the fiber $f^{-1}(y)$, considered as a subspace of X.*

(3.6.2). In the rest of the text, when we consider a fiber $f^{-1}(y)$ to be a prescheme, we are referring to the prescheme over $k(y)$ obtained by transporting the prescheme structure on $X \times_Y \mathrm{Spec}(k(y))$ onto $f^{-1}(y)$ via the homeomorphism given by the projection to X [Liu, 3.1.17].

We also write the above product as $X \otimes_Y k(y)$ or $X \otimes_{\mathcal{O}_y} k(y)$. More generally, if B is an $\mathcal{O}_y$-algebra, we denote $X \times_Y \mathrm{Spec}(B)$ by $X \otimes_Y B$ or $X \otimes_{\mathcal{O}_y} B$.

With these conventions, it follows from (3.5.10) that the points of X with values in an extension K of $k(y)$ are identified with the points of $f^{-1}(y)$ with values in K.

(3.6.3). Given $f\colon X \to Y$, $g\colon Y \to Z$, $h = g \circ f$, the fiber $h^{-1}(z)$ is isomorphic to

$$X \times_Z \mathrm{Spec}(k(z)) = X \times_Y g^{-1}(z).$$

In particular, if $U \subseteq X$ is open, then $U \cap f^{-1}(y)$, considered as an open sub-prescheme of $f^{-1}(y)$, is isomorphic to the fiber $(f_U)^{-1}(y)$, where f_U is the restriction of f to U.

Proposition (3.6.4). — *('Transitivity of fibers') Given $f\colon X \to Y$, $g\colon Y' \to Y$, let $X' = X_{(Y')}$ and $f' = f_{(Y')}$. For every $y' \in Y$, setting $y = g(y')$, the prescheme $f'^{-1}(y')$ is isomorphic to $f^{-1}(y) \times_{k(y)} k(y')$.*

In particular, if V is an open neighborhood of y and f_V denotes the restriction of f to $f^{-1}(V)$, then the preschemes $f^{-1}(y)$ and $f_V^{-1}(y)$ are canonically isomorphic.

Proposition (3.6.5). — *Let* $f\colon X \to Y$ *be a morphism,* y *a point of* Y*,* $Z = \mathrm{Spec}(\mathcal{O}_{Y,y})$*,* $p\colon X \times_Y Z \to X$ *the projection. Then* p *is a homeomorphism of* $X \times_Y Z$ *onto the subspace* $f^{-1}(Z)$ *of* X*, where* Z *is identified with a subspace of* Y *as in (2.4.2), and for every* $t \in X \times_Y Z$*, setting* $x = p(t)$*, the induced homomorphism* $p_t^\sharp\colon \mathcal{O}_x \to \mathcal{O}_t$ *is an isomorphism.*

3.7. **Application: reduction of a prescheme mod $\mathcal{I}$.**

[A footnote in EGA explains that this section depends on some results from later in Chapter I and Chapter II, is meant for readers familiar with classical algebraic geometry (before schemes), and will not be used elsewhere in EGA.]

(3.7.1). If X is an A-prescheme, and $\mathcal{I} \subseteq A$ is an ideal, then $X_0 = X \otimes_A (A/\mathcal{I})$ is an $(A/\mathcal{I})$-prescheme, said to be obtained from X by *reduction mod* $\mathcal{I}$.

(3.7.2). This terminology is used chiefly when A is local, $\mathcal{I}$ is maximal, and X_0 is therefore a prescheme over the residue field $k = A/\mathcal{I}$.

If A is an integral domain with fraction field K, we also have the K-prescheme $X' = X \otimes_A K$. By an abuse of language which we shall avoid, X_0 was traditionally said to be obtained also from X' by reduction mod $\mathcal{I}$. In the traditional situation, A was typically a discrete valation ring [for example, the ring of p-adic integers $\mathbb{Z}_{(p)}$, so that X' is a prescheme over $\mathbb{Q}$, while X_0 is a prescheme over the finite field $\mathbb{Z}/p\mathbb{Z}$], and X' was assumed to be a closed subscheme of an ambient prescheme P' such as projective space P^n_K, itself a base extension of a prescheme P over A, here $P = P^n_A$.

In this case, $Y = \mathrm{Spec}(A)$ has two points, the closed point $y = \mathcal{I}$, and the generic point (0), which is the unique point of an open set $U = \mathrm{Spec}(K)$ of Y. Then X' is just the open prescheme $\psi^{-1}(U) \subseteq X$ for the structure morphism $\psi\colon X \to Y$. In particular, a closed subscheme $X' \subseteq P'$ is a locally closed subscheme of P. Assuming P Noetherian, there is a smallest closed subscheme $X = \overline{X'} \subseteq P$ containing X', and $X \cap U = X'$. This allows us to regard X' as canonically of the form $X \otimes_A K$. Then the reduction $X_0 = X \otimes_A k$ of X modulo I is the fiber $\psi^{-1}(y)$ over the closed point $y \in Y$, regarded as a prescheme. Before schemes, the prescheme X was not considered explicitly, for lack of suitable terminology. However, classical results about X_0 are best understood as consequences of stronger results abour X.

(3.7.3). One particular fact which has tended to inhibit the conceptual clarification of this situation is that if A is a discrete valuation ring and X is *proper* over A (for instance if X is projective (II, 5.5.4)), then the points of X with values in A and the points of X' with values in K are in canonical bijection (II, 7.3.8). This sometimes leads results to be stated for X', which are really results about X, and which if stated in the latter form would be valid without the assumption that the local ring A is of dimension 1.