2. PRESCHEMES AND MORPHISMS OF PRESCHEMES

[Note on terminology: today the term scheme is usually used for what EGA calls a prescheme. What EGA calls a scheme is now called a separated scheme. Liu, in particular, uses the current terminology.]

2.1. Definition of preschemes.

(2.1.1). An open subset $V$ of a ringed space $X$ is called affine open if $(V, \mathcal{O}_X|_V)$ is an affine scheme (1.7.1).

Definition (2.1.2). — [Liu, 2.3.8] A prescheme is a ringed space $(X, \mathcal{O}_X)$ such that every point has an open affine neighborhood.

Proposition (2.1.3). — The open affine subsets of a prescheme form a base of its topology.

Proposition (2.1.4). — The underlying space of a prescheme is $T_0$.

Proposition (2.1.5). — Every irreducible closed subset of a prescheme $X$ has a unique generic point; thus $x \to \{x\}$ is a bijection from $X$ to its set of irreducible closed subsets [Liu, 2.4.12 is a special case].

(2.1.6). If $y$ is the generic point of an irreducible closed subset $Y \subseteq X$, we sometimes write $\mathcal{O}_{X/Y}$ for $\mathcal{O}_{X,y}$ and call it the local ring of $X$ along $Y$, or the local ring of $Y$ in $X$.

If $X$ is itself irreducible, with generic point $x$, then $\mathcal{O}_{X,x}$ is called the ring of rational functions on $X$.

Proposition (2.1.7). — [Liu, 2.3.9] If $X$ is a prescheme and $U \subseteq X$ is open, then $(U, \mathcal{O}_X|_U)$ is a prescheme.

This follows from (2.1.3).

(2.1.8). A prescheme $X$ is irreducible, or connected, if its underlying space is. $X$ is integral if it is irreducible and reduced (cf. (5.1.4)) [Liu, 2.4.16]. $X$ is locally integral if every $x \in X$ has an open neighborhood which is integral.

2.2. Morphisms of preschemes.

Definition (2.2.1). — [Liu, 2.3.13] A morphism of preschemes is a morphism of ringed spaces $(f, \phi): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $\phi^*_x: \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings for all $x \in X$.

In particular, $\phi^*_x$ induces a homomorphism $\phi^x: k(f(x)) \to k(x)$, making the field $k(x)$ an extension of $k(f(x))$.

[In other words, a morphism of preschemes $f: X \to Y$ is by definition a local morphism between the locally ringed spaces $X$, $Y$.]

(2.2.2). Morphism are closed under composition, making preschemes into a category.
Example (2.2.3). — If $U \subseteq X$ is open, the inclusion of $(U, \mathcal{O}_X|U)$ as an open sub-prescheme of $X$ is a morphism from $U$ to $X$. By (0, 4.1.1) this is a monomorphism in the category of ringed spaces and hence also in the category of preschemes.

Proposition (2.2.4). — [Liu, 2.3.25] Let $(X, \mathcal{O}_X)$ be a prescheme and $(S, \mathcal{O}_S) = \text{Spec}(A)$ an affine scheme. Then there is a canonical bijection between morphisms $X \to S$ and ring homomorphisms $A \to \mathcal{O}_X(X)$.

This holds more generally for any locally ringed space $(X, \mathcal{O}_X)$. See §1.8.

Proposition (2.2.5). — Let $f : X \to S$ correspond to $\phi : A \to \mathcal{O}_X(X)$ as in (2.2.4). Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a quasi-coherent sheaf of $\mathcal{O}_X$-modules (resp. $\mathcal{O}_Y$-modules), and let $M = \mathcal{F}(S)$ [so $\mathcal{F} = \mathcal{M}$]. Then $f$-morphisms $\mathcal{F} \to \mathcal{G}$ (0, 4.4.1) are in natural bijection with $A$-module homomorphisms $M \to \mathcal{G}(X)$.

(2.2.6). A morphism $(f, \phi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is said to be open if $f(U)$ is open for every open $U \subseteq X$, closed if $f(Z)$ is closed for every closed $Z \subseteq X$, dominant if $f(X)$ is dense in $Y$, surjective if $f$ is surjective. These conditions are properties of $f$ alone.

Proposition (2.2.7). — Let $X \to Y \to Z$ be morphisms of preschemes.

(i) If $f$ and $g$ are open (resp. closed, dominant, surjective), then so is $g \circ f$.

(ii) If $f$ is surjective and $g \circ f$ is closed, then $g$ is closed.

(iii) If $g \circ f$ is surjective, then $g$ is surjective.

Proposition (2.2.8). — Given a morphism $f : X \to Y$ and an open covering $Y = \bigcup_{\alpha} U_{\alpha}$, let $f_{\alpha} : f^{-1}(U_{\alpha}) \to U_{\alpha}$ be the restriction of $f$. Then $f$ is open (resp. closed, dominant, surjective) if and only if every $f_{\alpha}$ satisfies the same condition.

In other words, the conditions that $f$ is open, etc., are local on $Y$.

(2.2.9). Suppose $X$ and $Y$ have the same, finite, number of irreducible components $X_i, Y_i$, $1 \leq i \leq n$. Let $\xi_i$ (resp. $\eta_i$) be the generic point of $X_i$ (resp. $Y_i$). A morphism $(f, \phi) : X \to Y$ is called birational if $f^{-1}(\{\eta_i\}) = \{\xi_i\}$ and $\phi_{\xi_i}^f : \mathcal{O}_{\eta_i} \to \mathcal{O}_{\xi_i}$ is an isomorphism, for each $i$.

A birational morphism is dominant, hence surjective if it is closed.

(2.2.10). We often write just $f$ for a morphism $(f, \phi)$ and $U$ for an open subscheme $(U, \mathcal{O}_X|U)$.

2.3. Gluing preschemes.

(2.3.1). [Liu, 2.3.33] A ringed space constructed by gluing preschemes (0, 4.1.7) is again a prescheme. Every prescheme is a gluing of affine schemes.

Example (2.3.2). — [Liu, 2.3.34] Let $K$ be a field, $B = K[s]$, $C = K[t]$, $X_1 = \text{Spec}(B)$, $X_2 = \text{Spec}(C)$. Let $U_{12} = D(s) \subset X_1$, $U_{21} = D(t) \subset X_2$, so $U_{12} = \text{Spec}(K[s, s^{-1}])$, $U_{21} = \text{Spec}(K[t, t^{-1}])$. Let $u_{12} : U_{21} \to U_{12}$ correspond to the isomorphism $K[t, t^{-1}] \to K[s, s^{-1}]$ given by $t \mapsto 1/s$. Gluing $X_1$ and $X_2$ along $u_{12}$ gives the projective line $X = \mathbb{P}^1(K)$, a special case of a more general construction (II, 2.4.3). One proves that $\Gamma(X, \mathcal{O}_X) = K$, hence $X$ is not an affine scheme, as it would then be reduced to a point.
2.4. Local schemes.

(2.4.1) A local scheme is an affine scheme $X = \text{Spec}(A)$ where $A$ is a local ring. Then $X$ has a unique closed point $a$, and $a \in \{ b \}$ for all $b \in X$.

For any point $y$ of a prescheme $Y$, $\text{Spec}(\mathcal{O}_y)$ is called the local scheme of $Y$ at $y$. For any affine neighborhood $V = \text{Spec}(B)$ of $y$, we have $\mathcal{O}_y \cong B_y$, and $B \to B_y$ induces a morphism $\text{Spec}(\mathcal{O}_y) \to V$. Composing this with the inclusion $V \hookrightarrow Y$ gives a canonical morphism $\text{Spec}(\mathcal{O}_y) \to Y$ independent of the choice of $V$ [Liu, 2.3.16].

**Proposition (2.4.2).** Let $(f, \phi) : (\text{Spec}(\mathcal{O}_y), \mathcal{O}_y) \to (Y, \mathcal{O}_Y)$ be the canonical morphism. Then $f$ is a homeomorphism of $\text{Spec}(\mathcal{O}_y)$ onto the subspace $S_y \subseteq Y$ consisting of points $z$ such that $y \in \overline{\{ z \}}$, and if $z = f(p)$, then $\phi_z^\# : \mathcal{O}_z \to (\mathcal{O}_y)_p$ is an isomorphism. Hence $(f, \phi)$ is a monomorphism of ringed spaces.

In particular, $\text{Spec}(\mathcal{O}_y)$ is in bijection with the set of irreducible closed subsets of $Y$ that contain $y$.

**Corollary (2.4.3).** A point $y \in Y$ is the generic point of an irreducible component of $Y$ if and only if the maximal ideal of $\mathcal{O}_y$ is its unique prime ideal (in other words, $\mathcal{O}_y$ has Krull dimension zero).

**Proposition (2.4.4).** Let $X = \text{Spec}(A)$ be a local scheme, a its unique closed point, $Y$ a prescheme. Every morphism $f : X \to Y$ factors uniquely as $X \to \text{Spec}(\mathcal{O}_{f(a)}) \to Y$. This gives a bijective correspondence between morphisms $X \to Y$, and pairs consisting of a point $y \in Y$ and a local homomorphism of local rings $\mathcal{O}_y \to A$.

(2.4.5) If $K$ is a field, $\text{Spec}(K)$ has only one point. If $A$ is local with maximal ideal $m$, then every local homomorphism $A \to K$ factors as $A \to A/m \to K$. Hence morphisms $\text{Spec}(A) \to \text{Spec}(K)$ are in bijection with homomorphisms of fields $A/m \to K$.

Given a prescheme $Y$, a point $y \in Y$, and an ideal $a_y \subseteq \mathcal{O}_y$, composing the canonical morphisms $\text{Spec}(\mathcal{O}_y/a_y) \to \text{Spec}(\mathcal{O}_y) \to Y$ gives a canonical morphism $\text{Spec}(\mathcal{O}_y/a_y) \to Y$. In particular, for $a_y = m_y$, we get $\text{Spec}(k(y)) \to Y$.

**Corollary (2.4.6).** Let $X = \text{Spec}(K) = \{ \xi \}$, where $K$ is a field. Every morphism $u : X \to Y$ factors uniquely as $X \to \text{Spec}(k(u(\xi))) \to Y$. This gives a bijective correspondence between morphisms $X \to Y$, and pairs consisting of a point $y \in Y$ and a field extension $k(y) \hookrightarrow K$.

**Corollary (2.4.7).** The canonical morphism $\text{Spec}(\mathcal{O}_y/a_y) \to Y$ is a monomorphism of ringed spaces.

**Remark (2.4.8).** Let $X$ be a local scheme, with closed point $a$. The only affine open subset of $X$ containing $a$ is $X$ itself. Hence an invertible sheaf (0, 5.4.1) of $\mathcal{O}_X$-modules is necessarily trivial, i.e., isomorphic to $\mathcal{O}_X$. This property does not hold for a general affine scheme $\text{Spec}(A)$. If $A$ is a normal domain, it is equivalent to $A$ having unique factorization.

2.5. Preschemes over a prescheme.
Definition (2.5.1). — [Liu, 2.3.21] Fix a prescheme \( S \). A \textit{prescheme over} \( S \), or \( S \)-prescheme, is a prescheme \( X \) together with a morphism \( \phi: X \to S \). One says that \( S \) is the \textit{base prescheme}, and \( \phi \) is the \textit{structure morphism}. If \( S = \text{Spec}(A) \) one also calls \( X \) a \textit{prescheme over} \( A \), or an \( A \)-prescheme.

By (2.2.4), to give an \( A \)-prescheme it is equivalent to give a prescheme \( X \) whose structure sheaf \( \mathcal{O}_X \) is a sheaf of \( A \)-algebras [Liu, 2.3.26]. In particular, every prescheme is a \( \mathbb{Z} \)-prescheme in a unique way [Liu, 2.3.27].

If \( \phi(x) = s \), we say that the point \( x \in X \) \textit{lies over} \( s \in S \). If \( \phi \) is dominant (2.2.6), we say that \( X \) \textit{dominates} \( S \).

(2.5.2). Given \( S \)-preschemes \( X \) and \( Y \), a morphism \( u: X \to Y \) is a \textit{morphism of} \( S \)-\textit{preschemes}, or \( S \)-\textit{morphism}, if \( \phi' \circ u = \phi \), where \( \phi, \phi' \) are the structure morphisms of \( X \) and \( Y \). In particular, \( u \) maps points of \( X \) lying over \( s \in S \) to points of \( Y \) also lying over \( s \). \( S \)-preschemes and \( S \)-morphisms form a category. We write \( \text{Hom}_S(X,Y) \) for the set of \( S \)-morphisms \( X \to Y \). If \( S = \text{Spec}(A) \), we also use the term \( A \)-\textit{morphism}.

(2.5.3). If \( X \) is an \( S \)-prescheme and \( v: X' \to X \) is any morphism, the composite \( X' \to X \to S \) makes \( X' \) an \( S \)-prescheme. In particular, open subschemes of an \( S \)-prescheme are naturally \( S \)-preschemes.

If \( u: X \to Y \) is an \( S \)-morphism, then so is the restriction of \( u \) to any open \( U \subseteq X \). Conversely, given an open covering \( X = \bigcup \alpha U_\alpha \), and \( S \)-morphisms \( u_\alpha: U_\alpha \to Y \) which agree on every \( U_\alpha \cap U_\beta \), there is a unique \( S \)-morphism \( u: X \to Y \) such that every \( u_\alpha \) is the restriction of \( u \).

If \( U \) is an open subscheme of \( X \), and \( V \) is an open subscheme of \( Y \) containing \( u(U) \), then \( u: U \to V \) is an \( S \)-morphism.

(2.5.4). Given a morphism \( S' \to S \), the composite \( X \to S' \to S \) makes any \( S' \)-prescheme an \( S \)-prescheme. Conversely, if \( S' \) is an open subscheme of \( S \), and \( X \) is an \( S \)-prescheme such that the image of its structure morphism is contained in \( S' \), then \( X \) is also an \( S' \)-prescheme, and if \( Y \) is another \( S \)-prescheme with the same property, then any \( S \)-morphism \( X \to Y \) is also an \( S' \)-morphism.

(2.5.5). An \textit{S-section} of an \( S \)-prescheme \( X \) is an \( S \)-morphism \( S \to X \) [Liu, 2.3.28]. We denote the set of \( S \)-sections of \( X \) by \( \Gamma(X/S) \) [or by \( X(S) \), as in Liu].