Synopsis of material from EGA Chapter I, §1.4–1.8

1. Affine schemes (continued)

1.4. **Quasi-coherent sheaves on a prime spectrum.**

Theorem (1.4.1). — *Let $X = \mathrm{Spec}(A)$, $V \subseteq X$ open and quasi-compact, and $\mathcal{F}$ a sheaf of $(\mathcal{O}_X|V)$ modules. The following are equivalent:*

(a) There is an A module M such that $\mathcal{F} \cong \widetilde{M}|V$.

(b) V has an open covering by subsets of the form $V_i = D(f_i)$ such that $\mathcal{F}|V_i \cong \widetilde{M}_i$ for some A_{f_i} module M_i, for each i.

(c) $\mathcal{F}$ is quasi-coherent (0, 5.1.3).

(d) For every $f \in A$ such that $D(f) \subseteq V$, the following two conditions hold:

(d 1) for every section $s \in \mathcal{F}(D(f))$, there is some n such that $f^n s$ extends to a section of $\mathcal{F}$ on V;

(d 2) for every section $t \in \mathcal{F}(V)$ such that $t|_{D(f)} = 0$, there is some n such that $f^n t = 0$.

One also has the variant of (d) in which instead of assuming $D(f) \subseteq V$, we allow any $D(f)$ and replace $D(f)$ with $D(f) \cap V$ in (d 1-2).

Corollary (1.4.2). — *Every quasi-coherent sheaf on a quasi-compact open subset of X is the restriction of a quasi-coherent sheaf on X.*

Corollary (1.4.3). — *Every quasi-coherent $\mathcal{O}_X$ algebra is isomorphic to $\widetilde{B}$ for an A algebra B, and every quasi-coherent $\widetilde{B}$ module is isomorphic to $\widetilde{N}$ for some B module N.*

1.5. **Coherent sheaves on a prime spectrum.**

Theorem (1.5.1). — *Let $X = \mathrm{Spec}(A)$, where A is* Noetherian. *Let $V \subseteq \mathrm{Spec}(A)$ be open and $\mathcal{F}$ a sheaf of $(\mathcal{O}_X|V)$ modules. The following are equivalent:*

(a) $\mathcal{F}$ is coherent.

(b) $\mathcal{F}$ is quasi-coherent and finitely generated.

(c) $\mathcal{F} \cong \widetilde{M}|V$ for a finitely generated A module M.

Corollary (1.5.2). — *Under the hypotheses of (1.5.1), $\mathcal{O}_X$ is a coherent sheaf of rings.*

Corollary (1.5.3). — *Under the hypotheses of (1.5.1), every coherent sheaf on an open subset of X is the restriction of a coherent sheaf on X.*

Corollary (1.5.4). — *Under the hypotheses of (1.5.1), every quasi-coherent sheaf on X is the direct limit of coherent subsheaves.*

1.6. **Functorial properties of quasi-coherent sheaves on prime spectra.**

(1.6.1). Let $\phi\colon A' \to A$ be a ring homomorphism and

$$^a\phi\colon X = \mathrm{Spec}(A) \to X' = \mathrm{Spec}(A')$$

the associated continuous map (1.2.1). We define a canonical homomorphism of sheaves of rings

$$\widetilde{\phi}\colon \mathcal{O}_{X'} \to {}^a\phi_*\mathcal{O}_X$$

as follows. Given $f' \in A'$, let $f = \phi(f')$. Then ${}^a\phi^{-1}(D(f')) = D(f)$, $\Gamma(D(f'), \widetilde{A'}) = A'_{f'}$, and $\Gamma(D(f), \widetilde{A}) = A_f$. The canonical homomorphism $\phi_{f'}\colon A'_{f'} \to A_f$ is thus identified with a ring homomorphism

$$\Gamma(D(f'), \widetilde{A'}) \to \Gamma(D(f'), {}^a\phi_*\widetilde{A}).$$

These homomorphisms are compatible with restriction, and the sets $D(f')$ form a base of open sets, so this defines $\widetilde{\phi}$. Then $({}^a\phi, \widetilde{\phi})$ is a morphism of ringed spaces (0, 4.1.1) [cf. Liu, 2.3.14]

$$\Phi\colon (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'}).$$

Note that the stalk homomorphism $\widetilde{\phi}^\sharp_x$ (0, 3.7.1) is just the canonical homomorphism $\phi_x\colon A'_{x'} \to A_x$, where $x' = {}^a\phi(x)$ (0, 1.5.1).

Example (1.6.2). — [cf. Liu, 2.3.15] Consider $\phi\colon A \to S^{-1}A$. In (1.2.6), we saw that ${}^a\phi$ is a homeomorphism of $Y = \operatorname{Spec}(S^{-1}A)$ onto the subspace of points $x \in X = \operatorname{Spec}(A)$ such that $\mathfrak{i}_x \cap S = \emptyset$. In this case, for $x = {}^a\phi(y)$, the stalk homorphism $\widetilde{\phi}^\sharp_y\colon \mathcal{O}_x \to \mathcal{O}_y$ is an isomorphism (0, 1.2.6), which is to say, $\mathcal{O}_Y$ is the restriction of $\mathcal{O}_X$ to the subspace Y.

Proposition (1.6.3). — *For every A module M, there is a canonical functorial isomorphism $\Phi_*(\widetilde{M}) \cong (M_{[\phi]})^\sim$. If $M = B$ is an A algebra, this is an isomorphism of sheaves of $\mathcal{O}_{X'}$ algebras.*

Corollary (1.6.4). — *Φ_* is an exact functor on the category of quasi-coherent $\mathcal{O}_X$ modules [cf. Liu, 5.1.8].*

Proposition (1.6.5). — *Let N' be an A' module and set $N = N' \otimes_{A'} A$. Then there is a canonical functorial isomorphism $\Phi^*(\widetilde{N'}) \cong \widetilde{N}$.*

Corollary (1.6.6). — *The sections of $\Phi^*(\widetilde{N'})$ induced by global sections of $\widetilde{N'}$ generate $\Gamma(X, \Phi^*\widetilde{N'})$.*

(1.6.7). In the setting of (1.6.5), the canonical homomorphism $\widetilde{N'} \to \Phi_*\Phi^*(\widetilde{N'})$ is $\widetilde{j}$, where $j\colon N' \to N' \otimes_{A'} A_{[\phi]}$ is given by $z' \mapsto z' \otimes 1$. Similarly, the canonical homomorphism $\Phi^*\Phi_*(\widetilde{M}) \to \widetilde{M}$ is $\widetilde{p}$, where $p\colon M_{[\phi]} \otimes_{A'} A \to M$ is given by $z \otimes a \mapsto az$.

It follows that if $v\colon N' \to M_{[\phi]}$ is an A' module homomorphism, then $\widetilde{v}^\sharp = (v \otimes 1)^\sim$ [as homomorphisms from $\Phi^*\widetilde{N'} = (N' \otimes_{A'} A)^\sim$ to $\widetilde{M} = (M \otimes_A A)^\sim$].

(1.6.8). Let N'_1, N'_2 be A' modules, with N'_1 finitely presented. Then the canonical homomorphism

$$\Phi^*(\mathcal{H}om_{\mathcal{O}_{X'}}(\widetilde{N'_1}, \widetilde{N'_2})) \to \mathcal{H}om_{\mathcal{O}_X}(\Phi^*(\widetilde{N'_1}), \Phi^*(\widetilde{N'_2}))$$

is $\widetilde{\gamma}$, where γ is the canonical A module homomorphism $\operatorname{Hom}_{A'}(N'_1, N'_2) \otimes_{A'} A \to \operatorname{Hom}_A(N'_1 \otimes_{A'} A, N'_2 \otimes_{A'} A)$.

(1.6.9). Let $\mathcal{I}'$ be an ideal of A' and M an A module. Then $\widetilde{\mathcal{I}'}\widetilde{M}$ (which by definition means the image of $\Phi^*(\widetilde{\mathcal{I}'})\otimes_{\mathcal{O}_X}\widetilde{M} \to \widetilde{M}$) is identified canonically with $(\mathcal{I}'M)^{\sim}$. In particular, taking $M = A$ and using the right exactness of Φ^*, the sheaf of $\mathcal{O}_X$ algebras $\Phi^*((A'/\mathcal{I}')^{\sim})$ is identified with $(A/\mathcal{I}'A)^{\sim}$.

(1.6.10). Given a third ring A'' and $\phi' : A'' \to A'$, let $\phi'' = \phi \circ \phi'$. Then $\Phi'' = \Phi' \circ \Phi$, that is, $A \to (\mathrm{Spec}(A), \widetilde{A})$ is a contravariant functor from commutative rings to ringed spaces.

1.7. **Characterization of morphisms of affine schemes.**

Definition (1.7.1). — An *affine scheme* is a ringed space $(X, \mathcal{O}_X)$ isomorphic to $(\mathrm{Spec}(A), \widetilde{A})$ for a commutative ring A. Then $\Gamma(X, \mathcal{O}_X) \cong A$ by (1.3.7) and we call it the *ring of the affine scheme* $(X, \mathcal{O}_X)$. Sometimes we denote it $A(X)$.

By abuse of language, *the affine scheme* $\mathrm{Spec}(A)$ means the ringed space $(\mathrm{Spec}(A), \widetilde{A})$.

(1.7.2). Given $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and a ring homomorphism $\phi\colon B \to A$, we constructed in (1.6.1) the morphism of ringed spaces $\Phi = ({}^a\phi, \widetilde{\phi}) = \mathrm{Spec}(\phi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Note that Φ determines ϕ as $\phi = \Gamma(\widetilde{\phi})\colon \Gamma(\widetilde{B}) \to \Gamma({}^a\phi_*\widetilde{A}) = \Gamma(\widetilde{A})$.

Theorem (1.7.3). — *The necessary and sufficient condition for a morphism of ringed spaces* $(\psi, \theta)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *between affine schemes to be of the form* $({}^a\phi, \widetilde{\phi})$ *for a ring homomorphism* $\phi\colon A(Y) \to A(X)$ *is that for all* $x \in X$, $\theta^\sharp_x\colon \mathcal{O}_{\psi(x)} \to \mathcal{O}_x$ *is a local homomomorphism of local rings.*

[A ringed space $(X, \mathcal{O})$ is said to be *locally ringed* if every stalk $\mathcal{O}_x$ is a local ring. The condition on (ψ, θ) in the theorem is the definition of *morphism of locally ringed spaces*—see (1.8.2)]

We define a *morphism of affine schemes* to be a morphism of locally ringed spaces between affine schemes.

Corollary (1.7.4). — *If* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *are affine schemes then the set of morphisms of affine schemes* $\mathrm{Hom}(X, Y)$ *is in canonical bijection with the set of ring homorphisms* $\mathrm{Hom}(B, A)$, *where* $A = \Gamma(\mathcal{O}_X)$, $B = \Gamma(\mathcal{O}_Y)$.

1.8. **Morphisms from locally ringed spaces to affine schemes.**

[This section was added in the list of Errata and Addenda to Vol. I found at the end of Vol. II.]

Proposition (1.8.1). — *Let* $(S, \mathcal{O}_S)$ *be an affine scheme and* $(X, \mathcal{O}_X)$ *any locally ringed space. There is a canonical bijection between ring homomorphisms* $\Gamma(S, \mathcal{O}_S) \to \Gamma(X, \mathcal{O}_X)$ *and morphisms of ringed spaces* $(\psi, \theta)\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ *such that* $\theta^\sharp_x\colon \mathcal{O}_{\psi(x)} \to \mathcal{O}_x$ *is a local homomomorphism of local rings for all* $x \in X$.

The bijection in the direction

$$\rho\colon \mathrm{Hom}((X, \mathcal{O}_X), (S, \mathcal{O}_S)) \to \mathrm{Hom}(\Gamma(S, \mathcal{O}_S), \Gamma(X, \mathcal{O}_X)) \tag{1.8.1.1}$$

sends (ψ, θ) to $\Gamma(\theta)$.

(1.8.2). A morphism satisfying the condition in Proposition (1.8.1) is called a *morphism of locally ringed spaces*. With these morphisms, locally ringed spaces form a subcategory of the category of ringed spaces. Writing $\mathrm{Hom}_{\mathrm{rsp}}$ for morphisms in the full category of ringed spaces, and Hom for morphisms in the category of locally ringed spaces, (1.8.1.1) is a special case of a general functorial map

$$\rho\colon \mathrm{Hom}_{\mathrm{rsp}}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y)) \to \mathrm{Hom}(\Gamma(Y,\mathcal{O}_Y),\Gamma(X,\mathcal{O}_X)) \tag{1.8.2.1}$$

and its restriction to locally ringed spaces

$$\rho'\colon \mathrm{Hom}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y)) \to \mathrm{Hom}(\Gamma(Y,\mathcal{O}_Y),\Gamma(X,\mathcal{O}_X)). \tag{1.8.2.2}$$

Corollary (1.8.3). — *A locally ringed space $(Y,\mathcal{O}_Y)$ is an affine scheme if and only if the map ρ' in (1.8.2.2) is bijective for every locally ringed space $(X,\mathcal{O}_X)$.*

[Proposition (1.8.1) says that Spec(−) is right adjoint to the functor $X \to \Gamma(X,\mathcal{O}_X)$ from the category of locally ringed spaces to the opposite of the category of commutative rings. By (1.3.7), $\Gamma(-,\mathcal{O})$ is also left inverse to Spec(−), up to canonical isomorphism, making Spec(−) an equivalence from $(\mathrm{Rings})^{\mathrm{op}}$ onto its image, the full subcategory of affine schemes, in the category of locally ringed spaces. Corollary (1.8.2) further characterizes this image.]

(1.8.4). Let $S = \mathrm{Spec}(A)$ be an affine scheme. Let (S',A') be the ringed space in which S' is a single point, and A' is the unique sheaf with $\Gamma(S',A') = A$. Let $\pi\colon S \to S'$ be the unique map. Since $\Gamma(S,\mathcal{O}_S) = A$, the identity map on A defines a π-morphism $\iota\colon A' \to \mathcal{O}_S$, making $i = (\pi,\iota)\colon (S,\mathcal{O}_S) \to (S',A')$ a morphism of ringed spaces. Similarly, to any A module M corresponds an A' module M' with $\Gamma(S',M') = M$, and we have $i_*(\widetilde{M}) = M'$ (1.3.7).

Lemma (1.8.5). — *With the notation of (1.8.4), for every A module M, the canonical functorial homomorphism (0, 4.4.3.3)*

$$i^* i_*(\widetilde{M}) \to \widetilde{M} \tag{1.8.5.1}$$

is an isomorphism.

Corollary (1.8.6). — *Let $(X,\mathcal{O}_X)$ be a ringed space and $u\colon X \to S$ a morphism of ringed spaces. With the notation of (1.8.4), for every A module M, we have a canonical functorial isomorphism of $\mathcal{O}_X$ modules*

$$u^*(\widetilde{M}) \xrightarrow{\simeq} u^* i^*(M'). \tag{1.8.6.1}$$

Corollary (1.8.7). — *With the hypotheses of (1.8.6) for every A module M and $\mathcal{O}_X$ module $\mathcal{F}$, we have a canonical isomorphism, functorial in M and $\mathcal{F}$,*

$$\mathrm{Hom}_{\mathcal{O}_S}(\widetilde{M}, u_*(\mathcal{F})) \xrightarrow{\simeq} \mathrm{Hom}_{A'}(M', i_* u_*(\mathcal{F})). \tag{1.8.7.1}$$

[The right hand side is the same as $\mathrm{Hom}_A(M,\Gamma(X,\mathcal{F}))$.]

(1.8.8). With the notation of (1.8.4), to give a morphism of ringed spaces $X \to S'$ it is equivalent to give a ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus (1.8.1) can be interpreted as a bijection $\mathrm{Hom}(X, S) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{rsp}}(X, S')$. More generally for locally ringed spaces X and Y, taking (Y', A') the to be ringed space with one point and $\Gamma(Y', A') = \Gamma(Y, \mathcal{O}_Y)$, one can interpret (1.8.2.1) as a map

$$\rho\colon\ \mathrm{Hom}_{\mathrm{rsp}}(X, Y) \to \mathrm{Hom}_{\mathrm{rsp}}(X, Y') \tag{1.8.8.1}$$

and (1.8.3) as saying that affine schemes are characterized as those locally ringed spaces Y such that restriction

$$\rho'\colon\ \mathrm{Hom}(X, Y) \to \mathrm{Hom}_{\mathrm{rsp}}(X, Y') \tag{1.8.8.2}$$

of (1.8.8.1) is bijective for every locally ringed space Y. [This was suggested as a starting point for a theory of preschemes relative to any ringed space Z, to be developed in a future chapter, with the usual theory of preschemes over a ring A the special case $Z = (S', A')$. I'm not sure whether such a chapter was ever written.]

(1.8.9). Pairs $(X, \mathcal{F})$ where X is a locally ringed space and $\mathcal{F}$ is an $\mathcal{O}_X$ module form a category whose morphism are dihomomorphisms $(u, h)\colon (X, \mathcal{F}) \to (Y, \mathcal{G})$ in which u is a morphism of locally ringed spaces. Then there is a canonical functorial map

$$\mathrm{Hom}((X, \mathcal{F}), (Y, \mathcal{G})) \to \mathrm{Hom}((\mathcal{O}_X(X), \mathcal{F}(X)), (\mathcal{O}_Y(Y), \mathcal{G}(Y))), \tag{1.8.9.1}$$

the right hand side denoting the set of di-homomorphisms between pairs consisting of a ring and a module (0, 1.0.2).

Corollary (1.8.10). — *Let Y be a locally ringed space and $\mathcal{G}$ an $\mathcal{O}_Y$ module. The necessary and sufficient condition for Y to be an affine scheme and $\mathcal{G}$ a quasi-coherent module is that (1.8.9.1) is bijective for every locally ringed space X and $\mathcal{O}_X$ module $\mathcal{F}$.*

Remark (1.8.11). — The results in (1.7.3), (1.7.4) and (2.2.4) and the construction in (1.6.1) follow from (1.8.1); (1.6.3), (1.6.4) and (2.2.5) follow from (1.8.7), and (1.6.5) and (1.6.6) follow from (1.8.6).