Synopsis of material from EGA Chapter 0 (Vol. I) §1–4

1. Rings of fractions

1.0. **Rings and algebras.**

(1.0.1). All rings have a unit element 1. If we don't specify otherwise, rings are commutative and modules over non-commutative rings are left modules.

(1.0.2). Given a homomorphism of (possibly non-commutative) rings $\phi\colon A \to B$, every B module M has naturally an A module structure. We write $M_{[\phi]}$ instead of M if we want to make this explicit. Given an A module homomorphism $f\colon L \to M_{[\phi]}$, the pair (ϕ, f) is called a *di-homomorphism* form (A, L) to (B, M). This makes pairs (ring, module) the objects of a category.

(1.0.3). Given a (left) ideal $I \subseteq A$, one writes BI for the (left) ideal $B\phi(I) \subseteq B$, which is also the image of the canonical homomorphism $B \otimes_A I \to B$. Similarly for right ideals.

(1.0.4). Let A be commutative. An A algebra is a ring B with a homomorphism $\phi(A) \to B$ whose image is in the center of B. Then for every $I \subseteq A$, the ideal $IB = BI$ is a two-sided ideal of B, and if M is a B module, $IM = BIM$ is a submodule.

(1.0.5). An A algebra B is *integral* over A if each element $b \in B$ is the root of a monic polynomial over A; equivalently, b is contained in a subalgebra of B which is a finitely-generated A module. If B is commutative, this is equivalent to every finitely-generated subalgebra of B being a finitely-generated A module. So in this case, B is integral and of finite-type over A iff B is a f.-g. A module.

(1.0.6). A commutative ring A is an *integral domain* if products of non-zero elements are non-zero, including the empty product, *i.e.*, $1 \neq 0$. Thus the zero ring is not an integral domain. An ideal $\mathfrak{p}$ is prime iff $A/\mathfrak{p}$ is an integral domain. If A is not the zero ring it has at least one prime ideal [since maximal ideals are prime].

(1.0.7). A *local ring* A has a unique maximal proper ideal $\mathfrak{m}$. Equivalently, every $x \notin \mathfrak{m}$ is a unit in A. A homomorphism $\phi\colon A \to B$ of local rings $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ is *local* if $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$, which is equivalent to $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$. A local homorphism induces a homomorphism of residue fields $A/\mathfrak{m} \to B/\mathfrak{n}$. The composition of local homomorphisms is local.

1.1. **Radical of an ideal. Nilradical and radical of a ring.**

(1.1.1). The *radical* of an ideal $\mathfrak{a} \subseteq A$ is the ideal $\sqrt{\mathfrak{a}} = \{x \mid \exists n\, x^n \in \mathfrak{a}\}$. One has $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$, $\sqrt{(\mathfrak{a} \cap \mathfrak{b})} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$, and $\phi^{-1}(\sqrt{\mathfrak{a}}) \subseteq \sqrt{\phi^{-1}(\mathfrak{a})}$ for any ring homomorphism $\phi\colon A' \to A$.

If $\mathfrak{a} = \sqrt{\mathfrak{a}}$, we say that $\mathfrak{a}$ is a *radical ideal.* This is true iff $\mathfrak{a}$ is an intersection of prime ideals, and in general $\sqrt{\mathfrak{a}}$ is the intersection of the prime ideals $\mathfrak{p} \supseteq \mathfrak{a}$, or just of the minimal ones. If A is Noetherian there are finitely many minimal primes containing $\mathfrak{a}$.

The ideal $\sqrt{(0)}$ is called the *nilradical* of A. If $\sqrt{(0)} = (0)$, A is *reduced.* In other words, A has no non-zero nilpotent elements. $A/\mathfrak{a}$ is reduced iff $\mathfrak{a}$ is a radical ideal. A subring of a reduced ring is reduced.

(1.1.2). The (Jacobson) *radical* $\mathfrak{R}(A)$ of a (possibly noncommutative) ring A is the intersection of its maximal left ideals (equivalently, right ideals). The radical of $A/\mathfrak{R}(A)$ is zero.

1.2. **Modules and rings of fractions.**

(1.2.1). A subset $S \subseteq A$ is *multiplicative* if $1 \in S$ and S is closed under products. Important examples: (i) $S_f = \{f^n \mid n \geq 0\}$, (ii) $A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$.

(1.2.2). [cf. Liu p. 10] Given a multiplicative set S and an A module M, define $S^{-1}M$ to be the quotient of $M \times S$ by the equivalence relation

$$(m_1, s_1) \equiv (m_2, s_2) \quad \text{iff there exists } s \in S \text{ such that } s(s_1 m_2 - s_2 m_1) = 0.$$

Write m/s for the equivalence class of (m, s). One has a canonical map $i_M^S \colon M \to S^{-1}M$, $i(m) = m/1$. In general, i_M^S is neither surjective not injective. Its kernel is the set of elements $m \in M$ such that $sm = 0$ for some $s \in S$.

The usual arithmetic of fractions makes $S^{-1}A$ a ring, in which the elements $s/1$ are invertible, and $S^{-1}M$ an $S^{-1}A$ module. The canonical map i_A^S is a ring homomorphism, and the canonical map i_M^S is an A module homomorphism.

(1.2.3). When $S = S_f$, we write A_f, M_f instead of $S_f^{-1}A$, $S_f^{-1}M$. The ring A_f is isomorphic to $A[X]/(fX - 1)$. If f is a unit, then $A_f = A$, $M_f = M$; if f is nilpotent, then $A_f = \{0\}$, $M_f = \{0\}$.

When $S = A \setminus \mathfrak{p}$, we write $A_\mathfrak{p}$, $M_\mathfrak{p}$. In this case, $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p}A_\mathfrak{p}$ $(= S^{-1}\mathfrak{p})$, and $\mathfrak{p} = (i_A^S)^{-1}(\mathfrak{m})$. Passing to the quotient, i_A^S induces a ring homomorphism from A to the residue field $A_\mathfrak{p}/\mathfrak{m}$, which is identified with the field of fractions of the integral domain $A/\mathfrak{p}$.

(1.2.4). Universal property: any ring homomorphism $\phi \colon A \to B$ such that $\phi(S)$ consists of units in B factors uniquely through $i_A^S \colon A \to S^{-1}A$. Under the same hypotheses if N is a B module, any A module homomorphism $M \to N$ factors uniquely through $i_M^S \colon M \to S^{-1}M$.

(1.2.5). [cf. Liu 1.2.10] One has a canonical isomorphism $S^{-1}M \cong S^{-1}A \otimes_A M$ such that $m/s \leftrightarrow 1/s \otimes m$. More precisely, this gives a natural isomorphism between the functors $S^{-1}(-)$ and $S^{-1}A \otimes_A (-)$ from A modules to $S^{-1}A$ modules—see (1.3.1).

(1.2.6). For every ideal $\mathfrak{a}' \subseteq S^{-1}A$, $\mathfrak{a} = (i_A^S)^{-1}(\mathfrak{a}')$ is an ideal of A, and one has $\mathfrak{a}' = \mathfrak{a}S^{-1}A = S^{-1}\mathfrak{a}$. This gives a bijective, inclusion-preserving correspondence between the prime ideals of $S^{-1}A$ and those prime ideals $\mathfrak{p} \subseteq A$ such that $\mathfrak{p} \cap S = \emptyset$. The local rings $A_\mathfrak{p}$ and $(S^{-1}A)_{S^{-1}\mathfrak{p}}$ are canonically isomorphic.

(1.2.7). If A is an integral domain, then i_A^S is injective provided $0 \notin S$, and $S^{-1}A$ is a subring containing A of the fraction field K of A. In particular, $A_\mathfrak{p}$ is local with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, and $\mathfrak{p}A_\mathfrak{p} \cap A = \mathfrak{p}$.

(1.2.8). If A is reduced, then so is $S^{-1}A$.

1.3. **Functorial properties.**

(1.3.1–2). [cf. Liu 1.2.11] $M \mapsto S^{-1}M$ is an exact functor from A modules to $S^{-1}A$ modules. In particular, if $N, P \subseteq M$ are submodules, then so are $S^{-1}N, S^{-1}P \subseteq S^{-1}M$, and $S^{-1}(-)$ commutes with $\cap$ and $+$.

(1.3.3). The functor $S^{-1}(-)$, like all tensor products, commutes with direct limits.

(1.3.4). There is a natural isomorphism of functors

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \cong S^{-1}(M \otimes_A N)$$

such that $(m/s) \otimes (n/t) \leftrightarrow (m \otimes n)/st$.

(1.3.5). There is a natural transformation between functors

$$S^{-1}\operatorname{Hom}(M, N) \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

sending ψ/s to the homomorphism $m/t \mapsto \psi(m)/st$. If M is finitely presented (in particular, if A is Noetherian and M is finitely generated), this is an isomorphism.

1.4. **Change of multiplicative set.**

(1.4.1–2). Given multiplicative sets $S \subseteq T \subseteq A$, there is a canonical homomorphism $\rho^{S,T}\colon S^{-1}A \to T^{-1}A$, and for any A module M, a canonical homomorphism of $S^{-1}A$ modules $S^{-1}M \to T^{-1}M$ (under the identification $S^{-1}M = S^{-1}A \otimes_A M$, the latter is $\rho^{S,T} \otimes 1_M$). More precisely, we have a natural transformation of functors $S^{-1}(-) \to T^{-1}(-)$, and it commutes (in an appropriate sense) with the natural transformations in (1.3.4) and (1.3.5).

(1.4.3). If every element of T divides an element of S, then the transformation $\rho^{S,T}$ above is an isomorphism. Call S *saturated* if it contains all divisors of its elements. Then we can replace S by its saturation T and get essentially the same functor $S^{-1}(-) \cong T^{-1}(-)$.

(1.4.4). Given three multiplicative sets $S \subseteq T \subseteq U \subseteq A$, we have $\rho^{S,U} = \rho^{T,U} \circ \rho^{S,T}$.

(1.4.5). Given a filtered direct system of multiplicative sets $S_\alpha \subseteq A$, and S their union, there are canonical isomorphisms

$$\varinjlim S_\alpha^{-1}A \cong S^{-1}A, \quad \varinjlim S_\alpha^{-1}M \cong S^{-1}M,$$

the second one giving a natural isomorphism of functors.

(1.4.6). If S_1 and S_2 are multiplicative, then so is S_1S_2, and one has a natural isomorphism

$$S_1^{-1}(S_2^{-1}M) \cong (S_1S_2)^{-1}M$$

such that $(m/s)/t \leftrightarrow m/st$.

1.5. **Change of ring.**

(1.5.1). Given a ring homomorphism $\phi\colon A' \to A$ and multiplicative sets $S' \subseteq A'$, $S \subseteq A$ such that $\phi(S') \subseteq S$, one has by (1.2.4) a unique $\phi^{S'}\colon S'^{-1}A' \to S^{-1}A$ making a commutative diagram

$$\begin{array}{ccc} S'^{-1}A' & \xrightarrow{\phi^{S'}} & S^{-1}A \\ \uparrow & & \uparrow \\ A' & \xrightarrow{\phi} & A. \end{array}$$

If $\phi(S') = S$ and ϕ is surjective, then $\phi^{S'}$ is surjective.

(1.5.2). In the above setting, if M is an A module, it is also an A' module, and $S^{-1}M$ is also an $S'^{-1}A'$ module. There is a canonical homomorphism of $S'^{-1}A'$ modules

$$\sigma\colon S'^{-1}M \to S^{-1}M$$

sending m/s' to $m/\phi(s')$. This gives a natural transformation of functors, and if $\phi(S') = S$, it is an isomorphism. (1.4.1) is the special case $A' = A$. When $M = A$, $S'^{-1}A$ is a ring and $\sigma\colon S'^{-1}A \to S^{-1}A$ is a homomorphism of $S'^{-1}A'$ algebras.

(1.5.3). Composing the maps σ with those in (1.3.4) and (1.3.5) gives natural transformations

$$(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N) \leftarrow S'^{-1}(M \otimes_A N),$$
$$S'^{-1}\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

of which the first is an isomorphism if $\phi(S') = S$ and the second is an isomorphism if $\phi(S') = S$ and M is finitely presented.

(1.5.4–5). Consider the functor $- \otimes_{A'} A$ from A' modules to A modules [called *extension of scalars*, and left adjoint to the functor from A modules to A' modules sending M to $M_{[\phi]}$]. Given A' modules M', N', there are natural isomorphisms of $S^{-1}A$ modules

$$S^{-1}(N' \otimes_{A'} A) \cong (S'^{-1}N') \otimes_{S'^{-1}A} S^{-1}A,$$
$$S^{-1}(M' \otimes_{A'} N' \otimes_{A'} A) \cong (S'^{-1}M') \otimes_{S'^{-1}A} (S'^{-1}N') \otimes_{S'^{-1}A} S^{-1}A.$$

and a natural homomorphism

$$S^{-1}(\operatorname{Hom}_{A'}(M', N') \otimes_{A'} A) \to \operatorname{Hom}_{S'^{-1}A'}(S'^{-1}M', S'^{-1}N') \otimes_{S'^{-1}A'} S^{-1}A,$$

which is an isomorphism if M' is finitely presented. The last two of these follow from the first and (1.3.4-5).

(1.5.6). In the setting of (1.5.1), suppose we also have mutliplicative sets $T' \supseteq S'$, $T \supseteq S$ such that $\phi(T') \subseteq T$. Then the maps in (1.5.1-3) above are compatible with the maps $\rho^{S,T}$ in (1.4.1-2), in the sense that the obvious diagrams commute.

(1.5.7). Given a third ring A'' and a homomorphism $\phi'\colon A'' \to A'$, various natural compatibilities hold between the homomorphisms associated above with ϕ and those associated with ϕ' and $\phi \circ \phi'$.

(1.5.8). If A is a subring of B, then for every minimal prime ideal $\mathfrak{p} \subseteq A$ there exists a minimal prime $\mathfrak{q} \subseteq B$ such that $\mathfrak{p} = A \cap \mathfrak{q}$. Proof: by (1.3.2), $A_{\mathfrak{p}}$ is a subring of $B_{\mathfrak{p}}$. By (1.2.6), $\mathfrak{p}' = \mathfrak{p}A_{\mathfrak{p}}$ is the unique prime ideal of $A_{\mathfrak{p}}$. The ring $B_{\mathfrak{p}}$ is non-zero, hence has at least one prime ideal $\mathfrak{q}'$, and necessarily $\mathfrak{q}' \cap A_{\mathfrak{p}} = \mathfrak{p}'$. Let $\mathfrak{q}_1$ be the preimage in B of $\mathfrak{q}'$. Then $\mathfrak{q}_1 \cap A = \mathfrak{p}$, which forces $\mathfrak{q} \cap A = \mathfrak{p}$ for any minimal prime $\mathfrak{q}$ of B contained in $\mathfrak{q}_1$.

1.6. **The module M_f as a direct limit.**

(1.6.1–2). Given $f \in A$ and an A module M, consider the directed system

$$M_1 \underset{f}{\to} M_2 \underset{f}{\to} \cdots,$$

where every $M_n = M$, and the arrows are $m \mapsto fm$. There is a natural isomorphism

$$M_f \cong \varinjlim M_n.$$

Given another element $g \in A$, we have a homomorphism of directed systems

$$\begin{tikzcd} M_1 \arrow[r, "f"'] \arrow[d, "1"'] & M_2 \arrow[r, "f"'] \arrow[d, "g"'] & \cdots \arrow[r, "f"'] & M_n \arrow[r, "f"'] \arrow[d, "g^n"'] & \cdots \\ M'_1 \arrow[r, "fg"'] & M'_2 \arrow[r, "fg"'] & \cdots \arrow[r, "fg"'] & M'_n \arrow[r, "fg"'] & \cdots , \end{tikzcd}$$

where all the modules here are equal to M, and the induced map from $M_f = \varinjlim M_n$ to $M_{fg} = \varinjlim M'_n$ coincides with the map $\rho^{f,fg}$ given by (1.4.1) and (1.4.3) [note that S_f is contained in the saturation of S_{fg}].

1.7. **Support of a module.**

(1.7.1). The *support* $\mathrm{Supp}(M)$ of an A module M is the set of prime ideals $\mathfrak{p} \subseteq A$ such that $M_{\mathfrak{p}} \neq 0$. If $\mathrm{Supp}(M) = \emptyset$, then $M = 0$, since any prime containing the annihilator of a non-zero element of M belongs to $\mathrm{Supp}(M)$.

(1.7.2). If $0 \to N \to M \to P \to 0$ is an exact sequence, then $\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(P)$, by the exactness of localization (1.3.2).

(1.7.3). If M is the sum of some submodules M_λ, then $M_{\mathfrak{p}} = \sum_\lambda (M_\lambda)_{\mathfrak{p}}$ by (1.3.2–3), hence $\mathrm{Supp}(M) = \bigcup_\lambda \mathrm{Supp}(M_\lambda)$.

(1.7.4). If M is finitely generated, then $\mathrm{Supp}(M)$ is equal to the set of primes containing the annihilator of M, as one proves by using (1.7.3) to reduce to the case when M is generated by one element.

(1.7.5). If M and N are finitely generated, then $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. In particular, if M is a finitely generated A module and $\mathfrak{a} \subseteq A$ is an ideal, then $\mathrm{Supp}(M/\mathfrak{a}M)$ is the set of primes containing $\mathfrak{a} + \mathrm{ann}(M)$.

2. Irreducible spaces and Noetherian spaces

2.1. **Irreducible spaces.** [cf. Liu, Section 2.4.2]

(2.1.1). A topological space X is *irreducible* if it is non-empty and not a union of two distinct proper closed subsets. Equivalent formulations are $X \neq \emptyset$ along with any of (i) every two non-empty open subsets have non-empty intersection, (ii) every non-empty open subset is dense, (iii) every open subset is connected.

(2.1.2). A subspace Y of any space X is irreducible iff its closure $\overline{Y}$ is irreducible. In particular the closure of a point $\overline{\{x\}}$ is always irreducible. If $y \in \overline{\{x\}}$ (equivalently $\overline{\{y\}} \subseteq \overline{\{x\}}$) one says that y is a *specialization* of x, or x is a *generalization* of y. A point x such that $X = \overline{\{x\}}$ (if such a point exists) is a *generic point*. A generic point x is contained in every open subset $U \subseteq X$, and is a generic point of U.

(2.1.3). A space is T_0 (or 'Kolmogorov') if for every two distinct points x, y there is an open subset containing one but not the other. A generic point in an irreducible T_0 space is unique.

A space X is *quasi-compact* if every open cover of X has a finite sub-cover. Then every non-empty closed subset of X contains a minimal non-empty closed subset M. If X is also T_0, then M consists of a single point (referred to as a *closed point*).

(2.1.4). If X is irreducible, then so is any non-empty open subset $U \subseteq X$, and if X has a generic point, it is also a generic point of U.

If (U_α) is a non-empty covering of X by non-empty open subsets, then X is irreducible iff every U_α is irreducible and every $U_\alpha \cap U_\beta$ is non-empty.

(2.1.5). Let X be irreducible and $f\colon X \to Y$ continuous. Then $f(X)$ is irreducible, and if $x \in X$ is a generic point, then $f(x)$ is a generic point in $f(X)$ and also in $\overline{f(X)}$. If Y is irreducible and has a unique generic point y, then $f(X)$ is dense iff $f(x) = y$.

(2.1.6). Every irreducible subspace (and hence also every point) of any space X is contained in some *maximal* irreducible subspace, which is necessarily closed. The maximal irreducible subspaces of X are called its *irreducible components*. A generic point of an irreducible component is not contained in any other irreducible component. If X has only finitely many irreducible components Z_i, and we define $U_j = X \setminus \bigcup_{j \neq i} Z_i$, the sets U_j are open, irreducible, pairwise disjoint, and their union is dense in X.

If $U \subseteq X$ is open, the correspondence $Z \mapsto Z \cap U$ is a bijection from irreducible components of X which meet U, to irreducible components of U [cf. Liu, 2.4.5(b)].

(2.1.7). If X is a finite union of irreducible subspaces Y_i, then the irreducible components of X are the maximal members of the collection of the Y_i's [cf. Liu, 2.4.5(c)]. If $Y \subseteq X$ is a subspace which has finitely many irreducible components, then the closures of its components are the components of $\overline{Y}$.

(2.1.8). Let Y be irreducible with a unique generic point y. Let $f\colon X \to Y$ be continuous. If Z is an irreducible component of X which meets $f^{-1}(y)$, then $f(Z)$ is dense in Y, but not conversely. However, if Z has a generic point, then the converse holds. Moreover, if every irreducible component of X which meets $f^{-1}(y)$ has a generic point, then these components Z are in bijective correspondence with the components $Z \cap f^{-1}(y)$ of $f^{-1}(y)$, with Z and $Z \cap f^{-1}(y)$ having the same generic points.

2.2. **Noetherian spaces.**

(2.2.1). A space is *Noetherian* if A.C.C. holds for open subsets (equivalently, D.C.C. for closed subsets). It is *locally Noetherian* if every point has a neighborhood which is Noetherian.

(2.2.2). Principle of Noetherian induction: if E is a poset satifying D.C.C., and P is a property of elements of E satisfying the condition "$P(x)$ for all $x < a$ implies $P(a)$," then $P(a)$ holds for all $a \in E$.

(2.2.3). A subspace of a Noetherian space is Noetherian. If X is a finite union of Noetherian subspaces, then X is Noetherian.

(2.2.4). Every Noetherian space is quasi-compact. Conversely, if every open subset of X is quasi-compact, then X is Noetherian.

(2.2.5). A Noetherian space has only finitely many irreducible components. This follows easily by Noetherian induction. [cf. Liu, 2.4.9 for a special case.]

3. Supplement on Sheaves

[EGA presupposes some familiarity with sheaves of sets, abelian groups, rings and modules, citing as a reference R. Godement, Topologie Algébrique et Théorie des Faisceaux (Hermann, Paris 1958, 1964). In this section it is explained how the main concepts of sheaf theory make sense for sheaves taking values in any category K (subject in places to certain assumptions). This can also serve as an concise introduction to sheaves from first principles. For other introductions, see Liu 2.2, Eisenbud and Harris I.1.3, or Hartshorne II.1.]

3.1. **Sheaves with values in a category.**

(3.1.1–4). Suppose given a collection of objects A_α and morphisms $\rho_i\colon A_{\alpha_i} \to A_{\beta_i}$ in a category K. A *projective limit* of the system (A_α, ρ_i) is an object X, equipped with morphisms $\rho_\alpha\colon X \to A_\alpha$ commuting with all the morphisms ρ_i, such that for every such object X' with morphisms $\rho'_\alpha\colon X' \to A_\alpha$, there is a unique morphism $\phi\colon X' \to X$ such that $\rho'_\alpha = \rho_\alpha \circ \phi$ for all α. As with any object characterized by a universal property, a projective limit X is unique up to canonical isomorphism if it exists.

The open subsets of a topological space X are the objects of a category $\mathcal{U}$ with a unique morphism $U \to V$ whenever $V \subseteq U$. A *presheaf on X with values in a category K* is a functor $\mathcal{F}\colon \mathcal{U} \to K$. The morphisms $\rho^U_V\colon \mathcal{F}(U) \to \mathcal{F}(V)$ which $\mathcal{F}$ associates to inclusions $V \subseteq U$ are called *restriction morphisms*. $\mathcal{F}$ is a *sheaf* if it satisfies the axiom:

> (F) For every open cover $U = \bigcup_\alpha U_\alpha$, $\mathcal{F}(U)$ is the projective limit of the system given by the objects $\mathcal{F}(U_\alpha)$ and $\mathcal{F}(U_\alpha \cap U_\beta)$ and the restriction morphisms $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}$.

These definitions reduce to the usual ones [cf. Liu, 2.2.2 and 2.2.7] when K is the category of sets, or more generally when K is a category of sets equipped with algebraic structure, such as rings, abelian groups, etc.. In that case K admits all projective limits, which coincide with projective limits of the underlying sets.

Suppose, however, that K is for example the category of topological rings (with continuous ring homomorphisms). Then a sheaf $\mathcal{F}$ with values in K is a sheaf of rings in the usual sense, but with the additional requirement that for every open cover $U = \bigcup_\alpha U_\alpha$, the ring $\mathcal{F}(U)$ carries the coarsest topology such that all the restriction maps $\mathcal{F}(U) \to \mathcal{F}(U_\alpha)$ are continuous.

(3.1.5). [cf. Liu, 2.2.5] If $\mathcal{F}$ is a presheaf (resp. sheaf) with values in K and $U \subseteq X$ is an open set, the $\mathcal{F}(V)$ for open sets $V \subseteq U$ form a presheaf (resp. sheaf) on U, denoted $\mathcal{F}|U$. The restriction $\mathcal{F} \mapsto \mathcal{F}|U$ is a functor.

(3.1.6). If the category K admits *inductive limits* (the concept dual to that of projective limit defined above), one defines the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at $x \in X$ to be the inductive limit of the $\mathcal{F}(U)$ for all neighborhoods $x \in U$.

If K is a category of sets with algebraic structure (rings, abelian groups, …), then $\mathcal{F}_x$ is the direct limit, and we use the notation s_x for the germ in $\mathcal{F}_x$ of a section $s \in \mathcal{F}(U)$, where $x \in U$ [cf. Liu, 2.2.8]. The *support* of a sheaf of abelian groups, modules or rings is defined to be the set of points where its stalk is non-zero [cf. Liu, Exercise 2.2.5].

3.2. **Presheaves on a base of open sets.** [cf. Liu, 2.2.6]

[As used in EGA the term *base of open sets* for the topology on a space X means a collection $\mathfrak{B}$ of open sets such that (i) $\mathfrak{B}$ is closed under finite intersections and (ii) every open subset is a union of members of $\mathfrak{B}$.]

(3.2.1). Assume K admits all projective limits. Let $\mathfrak{B}$ be a base of open subsets on X. Regarding $\mathfrak{B}$ as a category with unique morphism $U \to V$ for $V \subseteq U$, a *presheaf on* $\mathfrak{B}$ *with values in* K is a contravariant functor $\mathcal{F}\colon \mathfrak{B} \to K$. For any open set U, the objects $\mathcal{F}(V)$ for $V \in \mathfrak{B}$ and $V \subseteq U$ form a projective system. Let $\mathcal{F}'(U) = \varprojlim \mathcal{F}(V)$ be its projective limit. Then $\mathcal{F}'$ is a presheaf on X with values in K, and for $U \in \mathfrak{B}$ one can identify $\mathcal{F}'(U)$ with $\mathcal{F}(U)$. (If X is a Noetherian space, it is possible to define $\mathcal{F}'(U)$ assuming only that K admits finite projective limits.)

(3.2.2). The necessary and sufficient condition for $\mathcal{F}'$ defined above to be a *sheaf* is:

> (F_0) For every $U \in \mathfrak{B}$ and open cover $U = \bigcup_\alpha U_\alpha$ with all $U_\alpha \in \mathfrak{B}$, $\mathcal{F}(U)$ is the projective limit of the system given by the objects $\mathcal{F}(U_\alpha)$ and $\mathcal{F}(U_\alpha \cap U_\beta)$ and the restriction morphisms $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}$.

A presheaf on $\mathfrak{B}$ is said to be a *sheaf* on $\mathfrak{B}$ if axiom (F_0) holds.

(3.2.3). A *morphism* $u\colon \mathcal{F} \to \mathcal{G}$ between presheaves on $\mathfrak{B}$ is a natural transformation of functors, *i.e.*, it consists of morphisms $u_V\colon \mathcal{F}(V) \to \mathcal{G}(V)$ for each $V \in \mathfrak{B}$, commuting with the restriction morphisms [cf. Liu, 2.2.10]. The construction of the presheaf $\mathcal{F}'$ on X from the presheaf $\mathcal{F}$ on $\mathfrak{B}$ is functorial.

(3.2.4). If K admits inductive limits, the stalk $\mathcal{F}'_x$ is canonically identified with the inductive limit $\varinjlim_{x \in V \in \mathfrak{B}} \mathcal{F}(V)$.

(3.2.5). If $\mathcal{F}$ is a sheaf on X and $\mathcal{F}_{\mathfrak{B}}$ its restriction to $\mathfrak{B}$, then $\mathcal{F} \cong \mathcal{F}'_{\mathfrak{B}}$ canonically. To give a morphism $u\colon \mathcal{F} \to \mathcal{G}$ it suffices to give $u_{\mathfrak{B}}\colon \mathcal{F}_{\mathfrak{B}} \to \mathcal{G}_{\mathfrak{B}}$.

(3.2.6). Given a projective system $(\mathcal{F}_\lambda)$ of sheaves with values in K, the presheaf $\mathcal{F}(U) = \varprojlim \mathcal{F}_\lambda(U)$ is a sheaf, and it is the projective limit of $(\mathcal{F}_\lambda)$ in the category of sheaves with values in K. If K is the category of sets, and $\mathcal{G}_\lambda \subseteq \mathcal{F}_\lambda$ is a system of subsheaves, then $\varprojlim \mathcal{G}_\lambda$ is a subsheaf of $\mathcal{F}$. For sheaves of abelian groups, the functor $\varprojlim$ is left exact.

3.3. **Gluing sheaves.**

(3.3.1). Assume K has all projective limits. Let X be a topological space and $(U_\lambda)_{\lambda \in L}$ an open covering. Suppose given for each λ a sheaf $\mathcal{F}_\lambda$ on U_λ with values in K, and for each λ, μ an isomorphism of sheaves $\theta_{\lambda\mu}\colon \mathcal{F}_\mu|(U_\lambda \cap U_\mu) \cong \mathcal{F}_\lambda|(U_\lambda \cap U_\mu)$. These are said to satisfy the *gluing condition* if for all λ, μ, ν the restrictions $\theta'_{\lambda\mu}$, $\theta'_{\mu\nu}$, $\theta'_{\lambda\nu}$ of $\theta_{\lambda\mu}$, $\theta_{\mu\nu}$, $\theta_{\lambda\nu}$ to $U_\lambda \cap U_\mu \cap U_\nu$ satisfy $\theta'_{\lambda\nu} = \theta'_{\lambda\mu} \circ \theta'_{\mu\nu}$. This is the necessary and sufficiient condition for there to exist a sheaf $\mathcal{F}$ on X and isomorphisms $\mathcal{F}|U_\lambda \cong \mathcal{F}_\lambda$ such their restrictions to each $U_\lambda \cap U_\mu$ commute with the given isomorphisms $\theta_{\lambda\mu}$.

Given any sheaf $\mathcal{F}$ on X and an open covering (U_λ), it is clear that $\mathcal{F}$ is the gluing of its restrictions $\mathcal{F}_\lambda = \mathcal{F}|U_\lambda$ along the identity maps $\mathcal{F}_\lambda|(U_\lambda \cap U_\mu) = \mathcal{F}_\mu|(U_\lambda \cap U_\mu)$.

(3.3.2). A system of morphisms $u_\lambda\colon \mathcal{F}_\lambda \to \mathcal{G}_\lambda$ which commute with the gluing isomoprhisms $\theta_{\lambda\mu}$ gives rise to a unique morphism $u\colon \mathcal{F} \to \mathcal{G}$ whose restriction to each U_λ is u_λ.

(3.3.3). The restriction $\mathcal{F}|V$ is the gluing of the restrictions $\mathcal{F}_\lambda|(V \cap U_\lambda)$.

3.4. **Direct images of presheaves.**

(3.4.1–2). Given a continuous map $f\colon X \to Y$ and a presheaf $\mathcal{F}$ on X, the *direct image* $f_*\mathcal{F}$ is the presheaf $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ [cf. Liu, p. 37]. If $\mathcal{F}$ is a sheaf, then so is $f_*\mathcal{F}$. The direct image f_* is a functor from presheaves (resp. sheaves) on X with values in K to presheaves (resp. sheaves) on Y with values in K.

(3.4.3). Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, we have $(g \circ f)_*\mathcal{F} = g_*f_*\mathcal{F}$. Given an open set $U \subseteq Y$, and setting $V = f^{-1}(U)$, we have $(f_*\mathcal{F})|U = (f|_V)_*(\mathcal{F}|V)$.

(3.4.4). Assume K admits inductive limits, so stalks make sense, and let $x \in X$, $y = f(x)$. Then there is a canonical morphism $f_x\colon (f_*\mathcal{F})_y \to \mathcal{F}_x$, functorial in $\mathcal{F}$. In general, f_x is *neither injective nor surjective*. Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, let $z = g(y)$; then $(g \circ f)_z = f_x \circ g_y$.

(3.4.5). If f is a homeomorphism of X onto $f(X)$, then f_x is an isomorphism. In particular, this applies to the inclusion $j\colon X \hookrightarrow Y$ of a subspace X of Y.

(3.4.6). If K is the category of groups, rings, etc., and $S \subseteq X$ is the support of $\mathcal{F}$, then the support of $f_*\mathcal{F}$ is contained in the closure $\overline{f(S)}$, but not necessarily in $f(S)$. In particular, if $j\colon X \hookrightarrow Y$ is a closed embedding, then the restriction of $j_*\mathcal{F}$ to $Y \setminus X$ is 0, but it may be non-zero if X is only locally closed.

3.5. **Inverse images of presheaves.**

[EGA uses $f^*\mathcal{G}$ to denote inverse image, but it is customary nowadays to write $f^{-1}\mathcal{G}$ instead and reserve the notation f^* for the inverse image of a sheaf of modules by a morphism of ringed spaces (see 4.3). I will follow current custom.]

(3.5.1). Given $f\colon X \to Y$ and presheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, a morphism $u\colon \mathcal{G} \to f_*\mathcal{F}$ is called an *f-morphism* from $\mathcal{G}$ to $\mathcal{F}$. For all open subsets $U \subseteq X$ and $f(U) \subseteq V \subseteq Y$, u induces morphisms $u_{U,V}\colon \mathcal{G}(V) \to \mathcal{F}(U)$, which commute with restriction to smaller open subsets U', V'. Conversely, any such family of morphisms $u_{U,V}$ commuting with restrictions determines an f-morphism $u\colon \mathcal{G} \to \mathcal{F}$.

If K admits all projective limits, and $\mathfrak{B}$, $\mathfrak{B}'$ are bases of the topologies on X and Y, it suffices to give the morphisms $u_{U,V}$ for $U \in \mathfrak{B}$, $V \in \mathfrak{B}'$.

If K admits inductive limits, then for each $x \in X$ and open neighborhood $f(x) \in V \subseteq Y$, we have a morphism $\mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)) \to \mathcal{F}_x$; in the limit these give a morphism $\mathcal{G}_{f(x)} \to \mathcal{F}_x$.

(3.5.2). Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, and f- and g-morphisms $u\colon \mathcal{G} \to f_*\mathcal{F}$, $v\colon \mathcal{H} \to g_*\mathcal{G}$, the composite $w\colon \mathcal{H} \underset{v}{\to} g_*\mathcal{G} \underset{g_*(u)}{\to} g_*f_*\mathcal{F}$ is a $(g \circ f)$-morphism. In this way, one can regard pairs $(X, \mathcal{F})$, where $\mathcal{F}$ is a presheaf on X with values in K, as forming a category, the morphisms $(X, \mathcal{F}) \to (Y, \mathcal{G})$ being pairs (f, u) consisting of a continuous map $f\colon X \to Y$ and an f-morphism $u\colon \mathcal{G} \to \mathcal{F}$.

(3.5.3). Given $f\colon X \to Y$ and a presheaf $\mathcal{G}$ on Y, an *inverse image by* f of $\mathcal{G}$ is a *sheaf* $\mathcal{G}'$ together with an f-morphism $\rho\colon \mathcal{G} \to \mathcal{G}'$ (*i.e.*, a presheaf homomorphism $\mathcal{G} \to f_*\mathcal{G}'$) such that for every *sheaf* $\mathcal{F}$ on X, the map

$$\mathrm{Hom}_X(\mathcal{G}', \mathcal{F}) \to \mathrm{Hom}_f(\mathcal{G}, \mathcal{F}) \underset{\mathrm{def}}{=} \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \tag{3.5.3.1}$$

induced by composition with ρ is bijective.

Since the pair $(\mathcal{G}', \rho)$ is characterized by a universal property, it is unique up to canonical isomorphism if it exists. Then we denote $\mathcal{G}' = f^{-1}\mathcal{G}$, $\rho = \rho_{\mathcal{G}}$ and call $f^{-1}\mathcal{G}$ *the inverse image sheaf* of $\mathcal{G}$ by f, equipped with the *canonical homomorphism* of presheaves

$$\rho_{\mathcal{G}}\colon \mathcal{G} \to f_* f^{-1}\mathcal{G}. \tag{3.5.3.2}$$

By definition, for any sheaf $\mathcal{F}$ on X, we have a bijective correspondence between homomorphisms of sheaves $v\colon f^{-1}\mathcal{G} \to \mathcal{F}$ on X and homomorphisms of presheaves $u\colon \mathcal{G} \to f_*\mathcal{F}$ on Y, the two being related by the fact that u factors as

$$u\colon \mathcal{G} \underset{\rho_{\mathcal{G}}}{\to} f_* f^{-1}\mathcal{G} \underset{f_*(v)}{\to} f_*\mathcal{F}. \tag{3.5.3.3}$$

(3.5.4). Suppose the category K is such that every presheaf $\mathcal{G}$ on Y admits an inverse image $f^{-1}\mathcal{G}$. One can show that this holds under quite general conditions on K; in particular it is true when K is the category of sets, abelian groups, or rings [in which case $f^{-1}\mathcal{G}$ coincides with the inverse image $f^{-1}\mathcal{G}$ discussed in Liu, p. 37 and Exercises 2.2.6, 2.2.13].

Then f^{-1} is a functor from presheaves on Y to sheaves on X, and the bijective correspondence in (3.5.3) is a functorial isomorphism

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}). \tag{3.5.4.1}$$

[In other words, although EGA does not use this language, (f^{-1}, f_*) is a pair of adjoint functors between sheaves on X and presheaves on Y.] The homomorphism correponding via (3.5.4.1) to $u\colon \mathcal{G} \to f_*\mathcal{F}$ is denoted $u^\sharp\colon f^{-1}\mathcal{G} \to \mathcal{F}$; inversely, the homomorphism corresponding to $v\colon f^{-1}\mathcal{G} \to \mathcal{F}$ is denoted $v^\flat\colon \mathcal{G} \to f_*\mathcal{F}$. In particular, the canonical homomorphism $\rho_{\mathcal{G}}$ in (3.5.3.2) is $i^\flat$, where i is the identity on $f^{-1}\mathcal{G}$, and (3.5.3.3) says that in general, $v^\flat = f_*(v) \circ \rho_{\mathcal{G}}$.

Similarly, given a sheaf $\mathcal{F}$ on X, there is a canonical homomorphism

$$\sigma_{\mathcal{F}}\colon f^{-1} f_*\mathcal{F} \to \mathcal{F}$$

given by $j^\sharp$ for j the identity on $f_*\mathcal{F}$. Then for $u \in \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$, we have $u^\sharp = \sigma_{\mathcal{F}} \circ f^{-1}(u)$.

[All this holds for any pair of adjoint functors. In current language, the canonical homomorphisms $\rho_{\mathcal{G}}$ and $\sigma_{\mathcal{F}}$, which are functorial in $\mathcal{G}$ and $\mathcal{F}$, respectively, are the *unit* and *co-unit* of the adjunction.]

(3.5.5). Given $X \underset{f}{\to} Y \underset{g}{\to} Z$, suppose that all presheaves on Y and on Z with values in K admit inverse images. Then there is a canonical natural isomorphism of functors $(g \circ f)^{-1} \cong f^{-1} \circ g^{-1}$.

(3.5.6). In the special case $f = 1_X \colon X \to X$, the inverse image $1_X^{-1}\mathcal{F}$ (when it exists) is the *sheaf associated to the presheaf* $\mathcal{F}$ [cf. Liu, 2.2.14]. Then every presheaf homomorphism from $\mathcal{F}$ to a *sheaf* $\mathcal{F}'$ factors uniquely through the canonical homomorphism $\mathcal{F} \to 1_X^{-1}\mathcal{F}$.

3.6. **Constant and locally constant sheaves.** [cf. Liu, 2.2.4, Exercise 2.2.1]

(3.6.1). A *constant presheaf* is a presheaf $\mathcal{F}$ such that the restriction morphism $\mathcal{F}(X) \to \mathcal{F}(U)$ is an isomorphism for all U. A *constant sheaf* is the sheaf associated to a constant presheaf. A sheaf $\mathcal{F}$ is *locally constant* if every $x \in X$ has a neighborhood U on which $\mathcal{F}|U$ is constant.

(3.6.2). If X is *irreducible* (not a union of two proper closed subsets), then the conditions

(a) $\mathcal{F}$ is a constant presheaf
(b) $\mathcal{F}$ is a constant sheaf
(c) $\mathcal{F}$ is a locally constant sheaf

are equivalent.

3.7. **Inverse images of presheaves of groups and rings.**

(3.7.1). Keep the notation of (3.5.3). When K is the category of sets, one can construct the inverse image $\mathcal{G}' = f^{-1}\mathcal{G}$ as follows. An element $s' \in \mathcal{G}'(U)$ is a family $(s'_x : x \in U)$, where $s'_x \in \mathcal{G}_{f(x)}$, and for every $x \in U$, the following condition holds: there is a neighborhood V of $f(x)$ in Y, a neighborhood $W \subseteq f^{-1}(V) \cap U$ of x, and a section $s \in \mathcal{G}(V)$ such that s'_z is the germ $s_{f(z)}$ for all $z \in W$. Informally, "s' is given locally by sections of $\mathcal{G}$." The restriction maps are the obvious ones, and the sheaf axiom for $\mathcal{G}'$ holds automatically, by the local nature of the construction.

One proves that $\mathcal{G}'$ as constructed above satisfies the universal property of $f^{-1}\mathcal{G}$. The description of f^{-1} as a functor is immediate: given a morphism $u \colon \mathcal{G}_1 \to \mathcal{G}_2$ and a section $s' = (s'_x) \in \mathcal{G}'_1$, define $f_*(u)(s') = (u_x(s'_x)) \in \mathcal{G}'_2$. When $f = 1_X$ we recover the standard construction of the sheaf associated to a presheaf of sets [cf. Liu, 2.2.15]. The preceding also applies verbatim to presheaves of groups and rings.

(3.7.2). In the setting of (3.7.1), if $\mathcal{G}$ is a *sheaf*, and $\rho \colon \mathcal{G} \to f_*f^{-1}\mathcal{G}$ is the canonical homomorphism, then the induced map on stalks $f_x \circ \rho_{f(x)}$ [see (3.4.4)] is a functorial isomorphism $\mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x$. It follows in particular that $\operatorname{Supp}(f^{-1}\mathcal{G}) = f^{-1}(\operatorname{Supp}(\mathcal{G}))$ and that the inverse image functor f^{-1} on sheaves of abelian groups is *exact*. [By definition the cokernel of a homomorphism of sheaves of abelian groups is the sheafification of its pre-sheaf cokernel, and this implies that a sequence of sheaves is exact if and only if it induces exact sequences on stalks.]

3.8. **Pseudo-discrete sheaves of topological spaces.**

As this section is used only for the construction of *formal schemes*, we omit it.

4. Ringed spaces

[Note: Liu (2.2.19) considers only *locally* ringed spaces, a restriction which is not needed or assumed for the general discussion in this section, although all (pre)schemes are in fact locally ringed spaces. Sheaves on locally ringed spaces are discussed in 5.5.]

4.1. **Ringed spaces, $\mathcal{A}$-modules and $\mathcal{A}$-algebras.**

(4.1.1). A *ringed space* (resp. *topologically ringed space*) $(X, \mathcal{A})$ is a topological space X with a sheaf of rings (resp. topological rings) $\mathcal{A}$. X is called the *underlying space* of $(X, \mathcal{A})$, and $\mathcal{A}$ its *structure sheaf*, also denoted $\mathcal{O}_X$. One often uses the abbreviation $\mathcal{O}_x$ for the stalk $\mathcal{O}_{X,x}$. We write 1 for the unit element in the ring of global sections $\mathcal{O}_X(X)$.

When not otherwise specified, $\mathcal{A}$ is assumed to be a sheaf of commutative rings.

Ringed spaces form a category. A morphism $(X, \mathcal{A}) \to (Y, \mathcal{B})$ is a pair (f, ϕ) where $f\colon X \to Y$ is a continuous map and $\phi\colon \mathcal{B} \to \mathcal{A}$ is an f-morphism (3.5.1), that is, a sheaf homomorphism $\phi\colon \mathcal{B} \to f_*\mathcal{A}$, which may also be specified by giving $\phi^\sharp\colon f^*\mathcal{B} \to \mathcal{A}$. [There is a notational conflict here with Liu, who writes $f^\sharp$ where Grothendieck writes ϕ.] Typically one abuses notation and writes f for the pair (f, ϕ).

The composition of $(f, \phi)\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ and $(g, \phi')\colon (Y, \mathcal{B}) \to (Z, \mathcal{C})$ is given by $(g \circ f, \phi'')$, where $\phi'' = g_*(\phi) \circ \phi'$ [see (3.5.2)]. This is equivalent to $\phi''^\sharp = \phi^\sharp \circ f^{-1}(\phi'^\sharp)$. Hence if $\phi^\sharp$ and $\phi'^\sharp$ are injective (resp. surjective), then so is $\phi''^\sharp$ (recall from (3.7.2) that f^{-1} is exact). If f is injective and $\phi^\sharp$ is surjective, then (f, ϕ) is a monomorphism in the category of ringed spaces.

(4.1.2). For any subset $M \subseteq X$, we have the ringed space $(M, \mathcal{A}|M)$, called the *restriction* of $(X, \mathcal{A})$ to M. Here $\mathcal{A}|M$ means $j^{-1}\mathcal{A}$ for the inclusion $j\colon M \to X$, generalizing the definition (3.1.5) in the case that M is open. The monomorphism of ringed spaces $(j, \omega)\colon (M, \mathcal{A}|M) \to (X, \mathcal{A})$, where $\omega^\sharp$ is the identity map on $\mathcal{A}|M$, is called the *canonical injection*. The composition of a morphism of ringed spaces $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ with (j, ω) is called the *restriction* of f to M.

(4.1.3). Recall the definition of a *sheaf of $\mathcal{A}$ modules*, or $\mathcal{A}$ *module* for short. [The definition, omittted in EGA, is a sheaf M of abelian groups, equipped with a homomorphism of sheaves of sets $\mathcal{A} \times M \to M$ making each $M(U)$ an $\mathcal{A}(U)$ module—cf. Liu, 5.1.1. A homomorphism of $\mathcal{A}$ modules is a homomorphism of sheaves of abelian groups $M \to N$ such that every $M(U) \to N(U)$ is an $\mathcal{A}(U)$ module homomorphism.] If $\mathcal{A}$ is non-commutative, we mean a sheaf of left $\mathcal{A}$ modules unless otherwise specified. A *sheaf of ideals in* $\mathcal{A}$ (left, right or two-sided) is an $\mathcal{A}$ submodule of $\mathcal{A}$.

Assuming $\mathcal{A}$ commutative, and replacing 'abelian group' and 'module' by 'ring' and 'algebra' in the definition of $\mathcal{A}$ module gives the definition of $\mathcal{A}$ *algebra*. Homomorphisms of $\mathcal{A}$ algebras are defined similarly. One can equivalently define an $\mathcal{A}$ algebra to be an $\mathcal{A}$ module $\mathcal{B}$ equipped with a homomorphism $\mu\colon \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ ('multiplication'), which is associative in the sense that the diagram below commutes.

$$\begin{array}{ccc} \mathcal{B}\otimes_{\mathcal{A}}\mathcal{B}\otimes_{\mathcal{A}}\mathcal{B} & \xrightarrow{\mu\otimes 1} & \mathcal{B}\otimes_{\mathcal{A}}\mathcal{B} \\ {\scriptstyle 1\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{B}\otimes_{\mathcal{A}}\mathcal{B} & \xrightarrow{\mu} & \mathcal{B}. \end{array}$$

The condition that $\mathcal{B}$ is commutative can similarly be expressed by a commutative diagram. [The tensor product of *presheaves* of $\mathcal{A}$ modules is defined by $(M\otimes_{\mathcal{A}}N)(U) = M(U)\otimes_{\mathcal{A}}N(U)$. If M and N are sheaves, their presheaf tensor product is not in general a sheaf. The tensor product of *sheaves* $M \otimes_{\mathcal{A}} N$ is defined to be the sheaf associated to the presheaf tensor product.]

If $\mathcal{M} \subseteq \mathcal{B}$ is an $\mathcal{A}$ submodule, the sum of the images of the $\mathcal{A}$ module homomorphisms $\bigotimes_{\mathcal{A}}^n \mathcal{M} \to \mathcal{B}$ for $n > 0$ is the $\mathcal{A}$ *subalgebra of* $\mathcal{B}$ *generated by* $\mathcal{M}$. It is also the sheaf associated to the presheaf which assigns to U the $\mathcal{A}(U)$ subalgebra of $\mathcal{B}(U)$ generated by $\mathcal{M}(U)$.

(4.1.4). A sheaf of rings $\mathcal{A}$ is *reduced at* $x \in X$ if the stalk $\mathcal{A}_x$ is reduced (1.1.1); *reduced* if it is reduced at every point. $\mathcal{A}$ is *regular* at x if $\mathcal{A}_x$ is a regular local ring [cf. Liu, 4.27]; *regular* if it is regular at every point. $\mathcal{A}$ is *normal* at x if $\mathcal{A}_x$ is an integrally closed domain [cf. Liu, proof of 4.1.21], *normal* if it is normal at every point. A ringed space $(X, \mathcal{A})$ is said to have any of these properties if $\mathcal{A}$ does.

A sheaf of rings $\mathcal{A}$ is *graded* if it is a direct sum of sheaves of abelian groups $\mathcal{A} = \bigoplus_n \mathcal{A}_n$, satisfying $\mathcal{A}_m\mathcal{A}_n \subseteq \mathcal{A}_{m+n}$. An $\mathcal{A}$ module $\mathcal{M}$ is graded if it is a direct sum of sheaves of abelian groups $\mathcal{M} = \bigoplus_n \mathcal{M}$, satisfying $\mathcal{A}_m\mathcal{M}_n \subseteq \mathcal{M}_{m+n}$. Clearly this makes each stalk $\mathcal{A}_x$ a graded ring, and $\mathcal{M}_x$ a graded module.

(4.1.5). Given a possibly non-commutative ringed space $(X, \mathcal{A})$, one has the bi-functors $\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}$ and $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ from sheaves of $\mathcal{A}$ modules (left or right, as appropriate) to sheaves of abelian groups, or more generally, to sheaves of $\mathcal{Z}$ modules, where $\mathcal{Z}$ is the center of $\mathcal{A}$. [For reference, here are the definitions: $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{A}(U)} \mathcal{G}(U)$; $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{A}|U}(\mathcal{F}|U, \mathcal{G}|U)$, which is already a sheaf by (3.3.2). See also Liu, p. 158 and Exercise 5.1.5(a).]

The stalk $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x$ is canonically isomorphic to $\mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$. There is a canonical homomorphism $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$, which is neither injective nor surjective in general.

The functor $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is right exact in each variable, and commutes with direct limits. The sheaves $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{F}$ and $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{A}$ are canonically isomorphic to $\mathcal{F}$.

The functors $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})(X)$ are left exact in each variable (these functors are contravariant in $\mathcal{F}$, so this means they take right exact sequences $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ to left exact sequences). The *dual* of a left $\mathcal{A}$ module $\mathcal{F}$ is the right $\mathcal{A}$ module $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})$ [cf. Liu, Exercise 5.1.12].

If $\mathcal{A}$ is commutative, the *p-th exterior power* $\bigwedge^p \mathcal{F}$ of an $\mathcal{A}$ module $\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto \bigwedge^p \mathcal{F}(U)$. The canonical homomorphism from this presheaf to its associated sheaf is injective. On stalks, we have $(\bigwedge^p \mathcal{F})_x = \bigwedge^p(\mathcal{F}_x)$. The exterior powers are functorial in $\mathcal{F}$.

(4.1.6). Let $\mathcal{A}$ be a possibly non-commutative sheaf of rings, $\mathcal{I}$ a sheaf of left ideals, $\mathcal{F}$ a left $\mathcal{A}$ module. Then $\mathcal{I}\mathcal{F}$ denotes the submodule of $\mathcal{F}$ which is the image of multiplication $\mathcal{I} \otimes_{\mathbb{Z}} \mathcal{F} \to \mathcal{F}$ (where $\mathbb{Z}$ is the constant sheaf associated to the presheaf $U \mapsto \mathbb{Z}$; if $\mathcal{A}$ is commutative we could also describe $\mathcal{I}$ as the image of $\mathcal{I} \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$). Clearly $(\mathcal{I}\mathcal{F})_x = \mathcal{I}_x \mathcal{F}_x$. It is also immediate that $\mathcal{I}\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{I}(U)\mathcal{F}(U)$, and that if $\mathcal{I}'$ is another sheaf of left ideals, then $\mathcal{I}(\mathcal{I}'\mathcal{F}) = (\mathcal{I}\mathcal{I}')\mathcal{F}$.

(4.1.7). Let $(X_\lambda, \mathcal{A}_\lambda)$ be a family of ringed spaces. For every two indices λ, μ suppose given an open set $V_{\lambda\mu} \subseteq X_\lambda$, and an isomorphism $\phi_{\lambda\mu} \colon (V_{\mu\lambda}, \mathcal{A}_\mu|V_{\mu\lambda}) \xrightarrow{\simeq} (V_{\lambda\mu}, \mathcal{A}_\lambda|V_{\lambda\mu})$, such that $V_{\lambda\lambda} = X_\lambda$ and $\phi_{\lambda\lambda}$ is the identity. Assume these data satisfy the *gluing condition*: for every three indices λ, μ, ν, if we denote the restriction of $\phi_{\lambda\mu}$ to $V_{\mu\lambda} \cap V_{\mu\nu}$ by $\phi'_{\lambda\mu}$, then $\phi'_{\lambda\mu}$ maps $V_{\mu\lambda} \cap V_{\mu\nu}$ onto $V_{\lambda\mu} \cap V_{\lambda\nu}$, and $\phi'_{\lambda\nu} = \phi'_{\lambda\mu} \circ \phi'_{\mu\nu}$.

Then one can construct a ringed space $(X, \mathcal{A})$ with a covering by open subsets X'_λ such that $(X'_\lambda, \mathcal{A}|X'_\lambda) \cong (X_\lambda, \mathcal{A}_\lambda)$, the sets $V_{\lambda\mu}$ and $V_{\mu\lambda}$ being identified with $X'_\lambda \cap X'_\mu$ so that the given isomorphism $\phi_{\lambda\mu}$ corresponds to the identity. The space $(X, \mathcal{A})$ is said to be constructed by *gluing the spaces* $(X_\lambda, \mathcal{A}_\lambda)$ *along the sets* $V_{\lambda\mu}$ by means of the maps $\phi_{\lambda\mu}$ [cf. Liu, 2.3.33].

4.2. **Direct image of an $\mathcal{A}$-module.**

(4.2.1). Given a morphism $(f, \phi) \colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ of ringed spaces and an $\mathcal{A}$ module $\mathcal{F}$, the direct image $f_*\mathcal{F}$ is naturally an $f_*\mathcal{A}$ module, and hence a $\mathcal{B}$ module via $\phi \colon \mathcal{B} \to f_*\mathcal{A}$. This makes f_* a left exact functor from $\mathcal{A}$ modules to $\mathcal{B}$ modules.

(4.2.2). There is a natural transformation of functors

$$f_*(\mathcal{F}) \otimes_{\mathcal{B}} f_*(\mathcal{G}) \to f_*(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}), \tag{4.2.2.1}$$

neither injective nor surjective in general, and a commutative diagram

$$\begin{array}{ccc} f_*(\mathcal{F}) \otimes_{\mathcal{B}} f_*(\mathcal{G}) \otimes_{\mathcal{B}} f_*(\mathcal{H}) & \longrightarrow & f_*(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}) \otimes_{\mathcal{B}} f_*(\mathcal{H}) \\ \big\downarrow & & \big\downarrow \\ f_*(\mathcal{F}) \otimes_{\mathcal{B}} f_*(\mathcal{G} \otimes_{\mathcal{A}} \mathcal{H}) & \longrightarrow & f_*(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G} \otimes_{\mathcal{A}} \mathcal{H}). \end{array} \tag{4.2.2.2}$$

(4.2.3). Similarly, there is a natural transformation

$$f_* \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{B}}(f_*\mathcal{F}, f_*\mathcal{G}). \tag{4.2.3.1}$$

(4.2.4). If $\mathcal{C}$ is an $\mathcal{A}$ algebra, then $f_*(\mathcal{C})$ is a $\mathcal{B}$ algebra, with multiplication defined by the composition

$$f_*(\mathcal{C}) \otimes_{\mathcal{B}} f_*(\mathcal{C}) \to f_*(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C}) \underset{f_*(\mu)}{\to} f_*(\mathcal{C}).$$

Associativity [see (4.1.3)] follows from (4.2.2.2). Similarly, if $\mathcal{F}$ is a $\mathcal{C}$ module, then $f_*(\mathcal{F})$ is naturally an $f_*(\mathcal{C})$ module.

(4.2.5). Consider the special case when $f \colon X \to Y$ is the inclusion of a *closed* subspace. Let $\mathcal{B}' = \mathcal{B}|X = f^{-1}\mathcal{B}$ be the restriction of $\mathcal{B}$ to X. An $\mathcal{A}$ module $\mathcal{M}$ on X can be considered as a $\mathcal{B}'$ module via $\phi^\sharp \colon \mathcal{B}' \to \mathcal{A}$. Then $f_*\mathcal{M}$ is the $\mathcal{B}$ module whose restriction to X is $\mathcal{M}$ and which is 0 outside of X. In this case, the natural transformations in (4.2.2) and (4.2.3) are isomorphisms.

(4.2.6). Given a third space $(Z, \mathcal{C})$ and a morphism $g\colon (Y, \mathcal{B}) \to (Z, \mathcal{C})$, the identity $(g \circ f)_* = g_* \circ f_*$ holds as an identity of functors from $\mathcal{A}$ modules to $\mathcal{B}$ modules.

4.3. **Inverse image of a $\mathcal{B}$-module.**

[As is nowadays customary, I abuse notation and write f for a morphism $(f, \phi)\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ of ringed spaces. We then write f^* for the preimage of a $\mathcal{B}$ module sheaf by the morphism (f, ϕ), as opposed to f^{-1} for the preimage of a general sheaf by the continuous map f, as in 3.5. In the EGA original, Grothendieck more correctly lets $F = (f, \phi)$, and writes F^*, f^* for our f^*, f^{-1}.]

(4.3.1). [cf. Liu, 5.1.13] Keep the notation of (4.2.1). The inverse image $f^{-1}(\mathcal{G})$ of a $\mathcal{B}$ module $\mathcal{G}$, constructed as in (3.7.1), is naturally an $f^{-1}(\mathcal{B})$ module. The homomorphism $\phi^\sharp\colon f^{-1}(\mathcal{B}) \to \mathcal{A}$ makes $\mathcal{A}$ an $f^{-1}(\mathcal{B})$ algebra. By extension of scalars, $f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{B})} \mathcal{A}$ is an $\mathcal{A}$ module, called the *inverse image of $\mathcal{G}$ by the morphism* (f, ϕ). We will denote it by $f^*(\mathcal{G})$. Then f^* is a *right exact* functor from $\mathcal{B}$ modules to $\mathcal{A}$ modules. It is not exact in general, since tensoring with $\mathcal{A}$ is only right exact.

One has $f^*(\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{B}_{f(x)}} \mathcal{A}_x$, by (3.7.2) [cf. Liu, 5.1.14].

(4.3.2). f^* commutes with direct limits, and hence with both finite and infinite direct sums.

(4.3.3). f^* commutes with tensor products, in the sense that one has a natural isomorphism

$$f^*(\mathcal{G}_1) \otimes_{\mathcal{A}} f^*(\mathcal{G}_2) \cong f^*(\mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2). \tag{4.3.3.1}$$

(4.3.4). If $\mathcal{C}$ is a $\mathcal{B}$ algebra, then $f^*(\mathcal{C})$ is naturally an $\mathcal{A}$ algebra. In particular, $f^*(\mathcal{B})$ is $\mathcal{A}$ itself. Likewise, if $\mathcal{M}$ is a $\mathcal{C}$ module, then $f^*(\mathcal{M})$ is an $f^*(\mathcal{C})$ module.

(4.3.5). If $\mathcal{I} \subseteq \mathcal{B}$ is a sheaf of ideals, then $f^{-1}(\mathcal{I})$ is a sheaf of ideals in $f^{-1}(\mathcal{B})$, and we have a canonical homorphism $f^*(\mathcal{I}) = f^{-1}(\mathcal{I}) \otimes_{f^{-1}(\mathcal{B})} \mathcal{A} \to \mathcal{A}$, whose image we denote by $f^*(\mathcal{I})\mathcal{A}$, or sometimes simply $\mathcal{I}\mathcal{A}$. Note that $\mathcal{I}\mathcal{A} = \phi^\sharp(f^{-1}(\mathcal{I}))\mathcal{A}$, and hence $(\mathcal{I}\mathcal{A})_x = \phi_x(\mathcal{I}_{f(x)})\mathcal{A}_x$. Given another ideal sheaf $\mathcal{I}' \subseteq \mathcal{B}$, we have $\mathcal{I}(\mathcal{I}'\mathcal{A}) = (\mathcal{I}\mathcal{I}')\mathcal{A}$.

If $\mathcal{F}$ is an $\mathcal{A}$ module, we define $\mathcal{I}\mathcal{F} = (\mathcal{I}\mathcal{A})\mathcal{F}$.

(4.3.6). Given a third space $(Z, \mathcal{C})$ and a morphism $(g, \phi')\colon (Y, \mathcal{B}) \to (Z, \mathcal{C})$, we have a canonical functorial isomorphism $(g \circ f)^* \cong f^* \circ g^*$.

4.4. **Relations between direct and inverse images.**

(4.4.1–3). [cf. Liu, Exercise 5.1.1] Keep the notation of (4.2.1). There is a canonical isomorphism of functors

$$\mathrm{Hom}_{\mathcal{A}}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{G}, f_*, \mathcal{F}), \tag{4.4.3.1}$$

i.e., (f^*, f_*) is a pair of adjoint functors between $\mathcal{B}$ modules and $\mathcal{A}$ modules. In particular, there are canonical homomorphisms

$$\rho_{\mathcal{G}}\colon \mathcal{G} \to f_* f^* \mathcal{G}, \tag{4.4.3.2}$$

$$\sigma_{\mathcal{F}}\colon f^* f_* \mathcal{F} \to \mathcal{F}, \tag{4.4.3.3}$$

which determine the isomorphism (4.4.3.1) in the same way as the corresponding maps for f^{-1} and f_* do in (3.5.3) and (3.5.4).

More explicitly, if s is a section of $\mathcal{G}$ on an open set $V \subseteq Y$, then $\rho_{\mathcal{G}}(s)$ is the section $s' \otimes 1$ of $f^*\mathcal{G}$ on $f^{-1}(V)$, where s' is given by $s'_x = s_{f(x)}$ for all $x \in f^{-1}(V)$.

Given a homomorphism $u\colon \mathcal{G} \to f_*\mathcal{F}$ and its corresponding homomorphism $u^\sharp\colon f^*\mathcal{G} \to \mathcal{F}$, one has homomorphisms $u_x\colon \mathcal{G}_{f(x)} \to \mathcal{F}_x$ on stalks, defined by composing $(u^\sharp)_x\colon (f^*\mathcal{G})_x \to \mathcal{F}_x$ with the canonical homomorphism $s_x \mapsto s_x \otimes 1$ from $\mathcal{G}_{f(x)}$ to $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{B}_{f(x)}} \mathcal{A}_x$. Equivalently, u_x is the direct limit of the homomorphisms $\mathcal{G}(V) \underset{u}{\to} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}_x$ over open neighborhoods V of $f(x)$.

(4.4.4). Given $u_1\colon \mathcal{G}_1 \to f_*\mathcal{F}_1$, $u_2\colon \mathcal{G}_2 \to f_*\mathcal{F}_2$, denote by $u_1 \otimes u_2$ the homomorphism $u\colon \mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2 \to f_*(\mathcal{F}_1 \otimes_{\mathcal{A}} \mathcal{F}_2)$ such that $u^\sharp = u_1^\sharp \otimes u_2^\sharp$ [this makes sense by (4.3.3)]. Then u is also the composite

$$\mathcal{G}_1 \otimes_{\mathcal{B}} \mathcal{G}_2 \overset{u_1 \otimes_{\mathcal{B}} u_2}{\to} (f_*\mathcal{F}_1) \otimes_{\mathcal{B}} (f_*\mathcal{F}_2) \to f_*(\mathcal{F}_1 \otimes_{\mathcal{A}} \mathcal{F}_2),$$

where the second arrow is given by (4.2.2.1).

(4.4.5). Let $(\mathcal{G}_\lambda)$ be a direct system of $\mathcal{B}$ modules and $u_\lambda\colon \mathcal{G}_\lambda \to f_*\mathcal{F}$ a system of homomorphisms commuting with the maps in $(\mathcal{G}_\lambda)$. Let $u = \varinjlim u_\lambda$ be the induced morphism from $\mathcal{G} = \varinjlim \mathcal{G}_\lambda$ to $f_*\mathcal{F}$. Then the homomorphisms $u_\lambda^\sharp\colon f^*\mathcal{G}_\lambda \to \mathcal{F}$ commute with the maps in the direct system $(f^*\mathcal{G}_\lambda)$ and we have $u^\sharp = \varinjlim u_\lambda^\sharp$.

(4.4.6). From the defimitions one obtains a natural transformation of functors

$$\gamma\colon\ \mathcal{H}om_{\mathcal{B}}(\mathcal{G}_1, \mathcal{G}_2) \to f_*(\mathcal{H}om_{\mathcal{A}}(f^*\mathcal{G}_1, f^*\mathcal{G}_2)),$$

and hence a corresponding canonical natural transformation

$$\gamma^\sharp\colon\ f^*\,\mathcal{H}om_{\mathcal{B}}(\mathcal{G}_1, \mathcal{G}_2) \to \mathcal{H}om_{\mathcal{A}}(f^*\mathcal{G}_1, f^*\mathcal{G}_2).$$

(4.4.7). If $\mathcal{F}$ is an $\mathcal{A}$ algebra, $\mathcal{G}$ is a $\mathcal{B}$ algebra, and $u\colon \mathcal{G} \to f_*\mathcal{F}$ is a $\mathcal{B}$ algebra homomorphism, then $u^\sharp\colon f^*\mathcal{G} \to \mathcal{F}$ is an $\mathcal{A}$ algebra homomorphism, and conversely.

(4.4.8). Given a third ringed space and morphism $(g, \phi')\colon (Y, \mathcal{B}) \to (Z, \mathcal{C})$, and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ modules $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, with homomorphisms $v\colon \mathcal{G} \to f_*\mathcal{F}$, $v'\colon \mathcal{H} \to g_*\mathcal{G}$, the composite $v'' = g_*(v) \circ v'\colon \mathcal{H} \to (g \circ f)_*\mathcal{F}$ corresponds to $(v'')^\sharp = v^\sharp \circ f^*(v'^\sharp)\colon (g \circ f)^*\mathcal{H} \to \mathcal{F}$.