MATH 256 HOMEWORK SET 8

1. Let $U_0, \dots, U_n$ be the standard covering of $\mathbb{P}^n_k$ by open affines. Describe the inclusion $j_i \colon \mathbb{A}^n \cong U_i \hookrightarrow \mathbb{P}^n_k$ in terms of the the functor on schemes over k represented by $\mathbb{P}^n_k$. What line bundle L on $\mathbb{A}^n$ and $n+1$ global sections generating L induce the morphism j_i?

2. If R is a graded ring, let $R^{(d)} = \bigoplus_n R_{dn}$.

(a) Prove (EGA II, 2.4.7(i)) that the inclusion $R^{(d)} \subseteq R$ induces an isomorphism $\operatorname{Proj}(R) \cong \operatorname{Proj}(R^{(d)})$. Your proof should also apply in the more general case where we allow R to be $\mathbb{Z}$ graded.

(b) Let R' be $R^{(d)}$ with the grading rescaled so that $R'_n = R_{dn}$. Assuming R_0 and R_1 generate R, show that R'_0 and R'_1 generate R', and that the twisting sheaf $\mathcal{O}'(1)$ on $Y = \operatorname{Proj}(R')$ coincides with $\mathcal{O}(d)$ for $Y = \operatorname{Proj}(R)$.

3. Let i be the Veronese embedding $i \colon \mathbb{P}^1_k \to \mathbb{P}^d_k$, the map induced by the complete linear system $\Gamma(\mathbb{P}^1, \mathcal{O}(d))$. Recall that if k is an algebraically closed field, i is given in coordinates by $(x : y) \to (x^d : x^{d-1}y : \cdots : y^d)$.

(a) Show that i also has the following description. Identify $\mathbb{P}^1_k = \operatorname{Proj}(k[x,y])$ with $\operatorname{Proj}(k[x,y]^{(d)})$ as in Problem 2. Then the surjective homomorphism $k[x_0, \dots, x_d] \to k[x,y]^{(d)}$ of graded rings sending x_i to $x^{n-i}y^i$ identifies $k[x,y]^{(d)}$ with $k[\boldsymbol{x}]/I$ for a graded ideal I and the Veronese map i with the closed immersion whose image is $V(\widetilde{I})$.

(b) Prove that I is the full ideal $\Gamma_\bullet(\mathbb{P}^n_k, \widetilde{I})$.

(c) Prove that I is generated by the quadratic polynomials $x_i x_j - x_k x_l$ such that $i+j = k+l$. Hint: let J be the ideal generated by these polynomials and prove that $k[\boldsymbol{x}]/J$ is generated as a k module by monomials whose images in $k[x,y]^{(d)}$ are linearly independent over k.

(d) Show that the Veronese map over k is a base extension of the Veronese map for $k = \mathbb{Z}$. Then define the Veronese map $\mathbb{P}^1_T \to \mathbb{P}^d_T$ over any scheme T.

4. (a) Consider the degree 2 Veronese map $i \colon \mathbb{P}^1_k \to \mathbb{P}^2_k$, whose image is the curve C in $\mathbb{P}^2$ defined by the graded ideal $I = (x_1^2 - x_0x_2, x_2^2 - x_1x_3, x_1x_2 - x_0x_3)$. Assume for simplicity that k is an algebraically closed field. What happens if you leave out the last generator $x_1x_2 - x_0x_3$ of I?

5. (a) Prove that if X is affine over S then every S-morphism $\mathbb{P}^n_S \to X$ factors as a section $S \to X$ of X over S composed with the structure morphisms $\mathbb{P}^n_S \to S$ (where $\mathbb{P}^n_S = S \times_{\operatorname{Spec}(\mathbb{Z})} \mathbb{P}^n_{\mathbb{Z}}$ by definition.) In particular, if $S = \operatorname{Spec}(k)$, where k is a field, then every k morphism from $\mathbb{P}^n_k$ to an affine k scheme X is constant, *i.e.*, it factors through the reduced one-point scheme $\operatorname{Spec}(k)$.

(b) Deduce that if k is a commutative ring (not the zero ring) and $n > 0$, then a vector bundle over $\mathbb{P}^n_k$ cannot be an affine scheme.

(c) Construct an example of an affine variety X and a morphism $\pi \colon X \to \mathbb{P}^n_k$ for some $n > 0$ such that X is an affine line bundle over $\mathbb{P}^n_k$, *i.e.*, $\mathbb{P}^n_k$ can be covered by open sets U

such that $\pi^{-1}(U)$ is isomorphic to $\mathbb{A}^1_U$ as a scheme over U. Why does this not contradict part (b)?

Hint on (c): $SL_2(k)$ acts on $\mathbb{P}^1_k$ by linear change of coordinates. Choosing a point in $\mathbb{P}^1_k$ and acting on it gives a morphism $SL_2(k) \to \mathbb{P}^1_k$. Construct X as the intersection of $SL_2(k)$ with a hyperplane in the space $\mathbb{A}^4_k$ of 2×2 matrices.

6. Let $X = \mathrm{Proj}(S)$, where $S = k[x, y, z]$ with $\deg(x) = \deg(y) = 1$, $\deg(z) = 2$. Show that the sheaf of modules associated to $S(1)$ is not locally free.

7. Prove that if $\mathrm{Proj}(R)$ is quasi-compact (for example if R is finitely generated as an algebra over R_0, although this condition is not necessary), then there exists a d such that $L = R(d)^{\sim}$ is locally free. In fact, show that $\mathrm{Proj}(R) \cong \mathrm{Proj}(S)$ for another graded ring S, where $S_0 = R_0$ and S_1 generate S, such that the twisting sheaf $\mathcal{O}(1)$ on $\mathrm{Proj}(S)$ coincides with L.

8. Let $A = k[t_1, t_2, \ldots]$ be a polynomial ring in infinitely many variables over a field k. Let $E \subseteq \mathbb{P}^1_A = \mathrm{Proj}(A[x, y])$ be a section of $\mathbb{P}^1_A$ over $\mathrm{Spec}(A)$ (for example, $E = V(y)$), and let $F \subseteq \mathbb{P}^1_A$ be the the fiber over the origin $V(t_1, t_2, \ldots)$ in $\mathrm{Spec}(A)$. Let $Y = E \cup F$ and let $\mathcal{F} = \mathcal{O}(-1) \otimes_{\mathcal{O}_{\mathbb{P}^1_A}} i_*\mathcal{O}_Y$, where $i\colon Y \hookrightarrow \mathbb{P}^1_A$ is the inclusion morphism.

(a) Show that the quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1_A$ is locally finitely generated.

(b) Show that $\mathcal{F} = \widetilde{M}$ for a finitely generated graded $A[x, y]$ module M. Using the fact that $\mathbb{P}^1_A$ is quasi-compact, this follows from (a), but I am asking you to find a suitable module M explicitly.

(c) Show that $\Gamma(\mathbb{P}^1_A, \mathcal{F})$ is isomorphic as an A module to the ideal $(t_1, t_2, \ldots)$. In particular, it is not finitely generated. Later we will see that if A is Noetherian, the A module of global sections of a locally finitely-generated quasi-coherent sheaf on $\mathbb{P}^n(A)$ is always finitely generated. This example shows that the Noetherian hypothesis is necessary.

(d) Show that the above construction also provides an example of a graded A algebra R, generated over A by finitely many elements of degree 1, and a d such that $\Gamma(Y, \mathcal{O}(d))$ is not a finitely generated A module, where $Y = \mathrm{Proj}(R)$.

9. Let S be a positively graded ring, $A = S_0$, so $X = \mathrm{Proj}(S)$ is a scheme over $Y = \mathrm{Spec}(A)$. Let I be the set of elements $a \in A$ such that S_+ is contained in the radical of the annihilator of a. Prove that I is an ideal, and that $V(I) \subseteq Y$ is the scheme-theoretic closed image of the structure morphism $X \to Y$.

10. (a) Let $R = k[y_0, y_1, y_2 \ldots]$ be a polynomial ring in infinitely many variables, $J \subseteq R$ the ideal generated by the homogeneous quadratic polynomials $y_i y_j - y_0 y_{i+j}$ for all i, j, and $S = R/J$, a graded k-algebra generated by S_1, but not finitely generated. Show that $\mathrm{Proj}(S) \cong \mathbb{A}^1_k = \mathrm{Spec}\, k[t]$, so $\mathbb{A}^1_k$ is isomorphic to a closed subscheme of $\mathbb{P}^\infty_k = \mathrm{Proj}(R)$.

(b) Prove that $\mathbb{A}^1_k$ is not projective over $\mathrm{Spec}(k)$.

Note that, at least if k is an algebraically closed field, it is reasonable to regard $\mathrm{Proj}(R)$ as an infinite dimensional projective space, and the above immersion as being given in coordinates by $t \mapsto (1 : t : t^2 : \cdots)$. The corresponding immersion $t \mapsto (1 : t : \cdots : t^{N-1})$ of $\mathbb{A}^1$ into $\mathbb{P}^N$ is not closed, but extends to a closed immersion $\mathbb{P}^1 \to \mathbb{P}^N$ given by $(s : t) \mapsto (s^N : s^{N-1}t : \cdots : t^N)$, mapping the point $(0 : 1)$ (at "$t = \infty$") to $(0 : 0 : \cdots : 1)$. Such an extension is of course not possible for the immersion into $\mathbb{P}^\infty$.

11. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$ modules on a scheme Y, and $\mathbb{P}(\mathcal{E})$ its projective bundle, considered as a scheme over Y with structure morphism $p\colon \mathbb{P}(\mathcal{E}) \to Y$. Let $\mathcal{F}$ be the kernel of the canonical surjection $p^*\mathcal{E} \to \mathcal{O}_P(1)$, and $Q = \mathbb{P}(\mathcal{F})$, a projective bundle over $\mathbb{P}(\mathcal{E})$ and thereby also a scheme over Y. Show that Q represents a functor which associates to any Y-scheme X (with structure morphism q) the set of pairs of quasi-coherent subsheaves $\mathcal{F}_2 \subseteq \mathcal{F}_1 \subseteq q^*\mathcal{E}$ such that $\mathcal{E}/\mathcal{F}_1$ and $\mathcal{F}_1/\mathcal{F}_2$ are both invertible. Generalize by constructing a scheme that represents the functor of flags $\mathcal{F}_l \subseteq \cdots \subseteq \mathcal{F}_1 = q^*\mathcal{E}$ such that each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is invertible, for any l.

12. Show that every *degree-2 hypersurface* $V(f) \in \mathbb{P}^3_{\mathbb{C}}$, where f is a homogeneous quadratic polynomial in 4 variables, is isomorphic to one of the following:
(i) A non-reduced scheme X such that $X_{\mathrm{red}} \cong \mathbb{P}^2_{\mathbb{C}}$,
(ii) A union of two projective planes $\mathbb{P}^2(\mathbb{C})$ intersecting along a line $\mathbb{P}^1(\mathbb{C})$, or
(ii) The projective closure of the cone $z^2 = xy$ in $\mathbb{A}^3$, or
(iii) $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.
To what extent does this classification depend on the ground field being the complex numbers?

13. Let X be the non-separated gluing of two copies of $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ (k a field) along the open set $D(x)$. (a) Classify the invertible sheaves $\mathcal{L}$ on X, up to isomorphism. Which ones are generated by their global sections? (b) For each $\mathcal{L}$ describe explicitly the open set $G(\varepsilon)$ and the morphism $G(\varepsilon) \to \operatorname{Proj}(S)$ induced as in (EGA II, 4.5.1).