Math 256 Homework Set 7

1. (a) Find an example of a morphism $X \to Y$ and a point $z \in X \times_Y X$ such that $p_1(z) = p_2(z)$, but z is not on the diagonal.

(b) Prove that in the above situation, the point z is not in the closure of the diagonal.

(c) Let X be a scheme. Suppose that for every two distinct points $x, y \in X$, there exists an open subset $U \subseteq X$ containing x and y, and an element $f \in \mathcal{O}_X(U)$ such that $V(f) = \{z \in U \mid f_z \in \mathfrak{m}_z\}$ contains exactly one of the points x, y. Prove that X is separated.

2. Let $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$ be the classical affine line over a field k. Let $Z_m = \operatorname{Spec} k[x]/(x^m)$, a closed subscheme of X supported at the origin, but non-reduced for $m > 1$, which we may think of as the infinitesimal neighborhood of order m around the origin. Consider the morphism $f\colon Y = \coprod_m Z_m \to X$ which is the inclusion morphism on each Z_m.

(a) Is f a quasi-compact morphism?

(b) Is f a quasi-separated morphism? Is it separated?

(c) Is $f_*\mathcal{O}_Y$ a quasi-coherent sheaf on X?

(d) Is $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ injective? Is its kernel a quasi-coherent ideal sheaf?

(e) Since X is an affine scheme, f is determined by the corresonding ring homomorphism $k[x] \to \Gamma(Z, \mathcal{O}_Z)$. Is this ring homomorphism injective?

(f) Describe the scheme-theoretic image closure $\overline{f(Y)}$, that is, the smallest closed subscheme of X through which f factors.

(g) What is the image of the map f on underlying spaces and what is its closure in the topological space X?

3. Let I be the ideal in the polynomial ring $k[a_{1,1}, \ldots, a_{m,n}]$ generated by all 2×2 minors of the $m \times n$ matrix with entries $a_{i,j}$. Let $X = V(I)$, a closed subscheme of $\mathbb{A}^{mn}_k$.

(a) An $m \times n$ matrix of rank ≤ 1 over a field K is the product of a column vector in K^m times a row vector in K^n. Construct, for every ring k, a morphism $f\colon \mathbb{A}^m_k \times_k \mathbb{A}^n_k \to X$ which specializes, in the case that k is an algebraically closed field K, to this parametrization of of matrices of rank ≤ 1.

(b) Prove that the ring homomorphism ϕ corresponding to f is injective, by finding a set of monomials in the a_{ij} which generate $k[a_{1,1}, \ldots, a_{m,n}]/I$ as a k module and are mapped by ϕ to elements of $k[x_1, \ldots, x_m, y_1, \ldots, y_m]$ linearly independent over k.

(c) Deduce that if k is a reduced ring, then I is a radical ideal.

(d)* Use a similar technique to prove that the ideal generated by all $r \times r$ minors is a radical ideal.

4. Let $f\colon X \to Y$ be a morphism of separated schemes such that the scheme-theoretic closed image $\overline{f(X)}$ is equal to Y. Prove that f is an epimorphism in the category of *separated* schemes. More generally, show that for any morphisms $g_1, g_2\colon Y \to Z$ with Z separated, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$, even if you do not assume X, Y separated.

5. (a) Let $f\colon X \to Y$ be a morphism such that $\overline{f(X)} = Y$, and the only open subset of Y that contains $f(X)$ is Y itself. Prove that f is an epimorphism in the category of all (possibly non-separated) schemes.

(b) Show that if Y is reduced and locally of finite type over a field, then the obvious morphism from $X = \coprod_{y \in Y_{\mathrm{cl}}} \mathrm{Spec}(k(y))$ to Y is an epimorphism.

6. Let k be an algebraically closed field of characteristic not equal to 2. The map $(x_1, \ldots, x_n) \mapsto -(x_1, \ldots, x_n)$ generates an action of the two-element group $G = \{\pm 1\}$ on $\mathbb{A}^n_k$ and on its coordinate ring $k[x_1, \ldots, x_n]$. Let R be the ring of invariants. We can think of $\mathrm{Spec}(R)$ as the quotient $\mathbb{A}^n_k/G$.

(a) Describe R.

(b) Show that for every closed point $x \in \mathbb{A}^n_k/G$ other than the image of the origin, the scheme-theoretic fiber of the morphism $\pi\colon \mathbb{A}^n_k \to \mathbb{A}^n_k/G$ over x is a reduced subscheme, and these subschemes are exactly the G orbits of closed points in $\mathbb{A}^n$ other than the origin.

(c) Show that the scheme-theoretic fiber of π over the image x of the origin is a non-reduced subscheme of $\mathbb{A}^n_k$ whose underlying set consists only of the origin, and whose coordinate ring S has length (that is, dimension as a k vector space) $n+1$. Also show that S is the direct sum of a 1-dimensional subspace on which G acts trivially, and an n dimensional subspace on which the generator of G acts as multiplication by -1.

7. Consider the functor from sets to schemes over S which sends a set X to the disjoint union $\coprod_{x \in X} S$ of copies of S indexed by the elements of X. It is convenient to use the notation $X \times S = \coprod_{x \in X} S$.

(a) Show that there is a functorial isomoprhism $(X \times Y) \times S \cong (X \times S) \times_S (Y \times S)$.

(b) Show that if G is a group, then $G \times S$ is naturally a group scheme over S, in such a way that to give an action of the group scheme $G \times S$ on a scheme Y over S it is equivalent to give an action of the abstract group G by S-automorphisms of Y.

(c) In the situation of (b), show that the quotient $(G \times S)\backslash Y$, in the sense of the coequalizer of the action and the projection in the category of ringed spaces, coincides with the abstract quotient $\pi\colon Y \to G\backslash Y$, defined as the quotient topological space (the set of G orbits, with the toplogy in which U is open if and only if $\pi^{-1}(U)$ is open), equipped with the sheaf $\mathcal{O} = (\pi_* \mathcal{O}_Y)^G$ of G invariant sections of $\pi_* \mathcal{O}_Y$.

8. (a) Show that the quotient variety considered in Problem 6 is the quotient in the sense of Problem 7.

(b)* Prove more generally that if R is a finitely generated algebra over a field k (so $X = \mathrm{Spec}(R)$ is a classical affine variety), and G is a finite group acting by k-algebra automorphisms of R, then $\mathrm{Spec}(R^G)$ is the quotient $G\backslash X$ in the sense of Problem 7.

9. Prove that the morphism of schemes $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ corresponding to an algebraic extension of fields $K \subseteq L$ is universally injective (and hence universally bijective) if and only if the extension is purely inseparable.

10. Let X be a scheme and $f(t)$ a polynomial in one variable with coefficients in $\mathcal{O}_X(X)$. Let $\mathcal{A} = \mathcal{O}_X[t]/(f(t))$, a quasi-coherent sheaf of $\mathcal{O}_X$ algebras. Then $Y = \underline{\mathrm{Spec}}(\mathcal{A})$, a scheme affine over X, is the closed subscheme $V(\mathcal{I}) \subseteq \mathbb{A}^1_X = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{A}^1_{\mathbb{Z}}$, where $\mathcal{I}$ is the ideal sheaf generated by the global function $f(t)$ on $\mathbb{A}^1_X$. In particular, the structure morphism $Y \to X$ is separated.

(a) Suppose that $f(t)$ and $f'(t)$ generate the unit ideal sheaf in $\mathcal{O}_X[t]$ (note that the derivative of a polynomial makes sense formally with coefficients in any commutative ring). Prove that the diagonal $\Delta(Y) \subseteq Y \times_X Y$ is then open, as well as closed. Hint: working on an affine $U = \mathrm{Spec}(R) \subseteq X$, relate the polynomial $(f(s) - f(t))/(s - t) \in R[s,t]$ to the complement of $\Delta(Y)$ and to $f'(t)$.

(b) Prove that the condition on f in (a) is equivalent to the following: for every $x \in X$, if k_x is its residue field, then the image of $f(t)$ in $k_x[t]$ is non-zero and has no multiple roots (in the algebraic closure $\overline{k_x}$). In other words, the fiber of $Y \to X$ over every geometric point $\mathrm{Spec}(K) \to X$ is a reduced, finite subscheme of the affine line $\mathbb{A}^1_K$.

(c) Prove that if X has a covering by affines U such that the closed points of U are closed points of X (e.g., if X is affine, or Noetherian, or Jacobson), then in (b) it suffices for the condition to hold at closed points $x \in X$.

Remark: the conclusion in (a) is equivalent to $f\colon Y \to X$ being smooth of relative dimension 0, or *étale*. It turns out that every étale morphism is locally of the type described in this problem.