Math 256 Homework Set 6

1. (a) Show that a finite disjoint union of affine schemes $X = X_1 \sqcup \cdots \sqcup X_n$ is affine and describe its coordinate ring in terms of those of the X_i.

(b) Show that an infinite disjoint union of (non-empty) affine schemes is not affine in general.

2. (a) Let $(R_i)_{i\in I}$ be a directed system of commutative k algebras and let $R = \varinjlim_i R_i$ be their direct limit. Prove that $\operatorname{Spec}(R)$ is the projective limit of the schemes $X_i = \operatorname{Spec}(R_i)$ in the category of schemes over k.

(b) Prove that infinite products of affine schemes over k exist and are affine.

(c) Prove that if $\mathfrak{p}$ is a prime ideal of a ring A, the local scheme $\operatorname{Spec}(A_{\mathfrak{p}})$ is the product over $X = \operatorname{Spec}(A)$ of all standard affine open neighborhoods $X_f = \operatorname{Spec}(A_f)$ of $\mathfrak{p}$ in $\operatorname{Spec}(A)$. Show that it is also the product of *all* open neighborhoods of $\mathfrak{p}$ in $\operatorname{Spec}(R)$.

(d) What goes wrong if you try to use (b) and gluing to construct infinite products of arbitary schemes over a scheme S? Hint: you can see the difficulty by considering the problem of constructing the product over S of infinitely many arbitrary open subschemes $U_\alpha \subseteq S$.

(e)* Construct an example of an infinite collection of schemes over a scheme S for which the product does not exist in the category of schemes over S, and prove it.

3. Let $\mathcal{F}$ be a sheaf on a space X and let $(\mathcal{F}_\alpha)_{\alpha\in I}$ be a collection of subsheaves of $\mathcal{F}$, that is, sub-presheaves which are sheaves.

(a) Prove that the presheaf intersection $\bigcap_\alpha \mathcal{F}_\alpha$ is a subsheaf of $\mathcal{F}$.

(b) Show that if the collection is finite, the stalks of the intersection are given by

$$\left(\bigcap_\alpha \mathcal{F}_\alpha\right)_x = \bigcap_\alpha \mathcal{F}_{\alpha,x},$$

but that this does not hold in general for infinite intersections.

(c) Show that the presheaf union $\bigcup_\alpha \mathcal{F}_\alpha$ need not be a sheaf, even in the finite case.

(d) Prove that the stalks of the presheaf union are given by

$$\left(\bigcup_\alpha \mathcal{F}_\alpha\right)_x = \bigcup_\alpha \mathcal{F}_{\alpha,x},$$

even in the infinite case, and deduce that the sheafification of the presheaf union is a subsheaf of $\mathcal{F}$ with these stalks.

4. (a) Let k be a field and $\overline{k}$ its algebraic closure. Let K be a finite algebraic extension of K. Prove that $\operatorname{Spec}(\overline{k}) \times_{\operatorname{Spec}(k)} \operatorname{Spec}(K)$ has a finite number of points, that the Galois group G of K over k acts freely on them (that is, each point has trivial stabilizer), and the the G orbits correspond bijectively to intermediate fields $k \subseteq L \subseteq \overline{k}$ which are isomorphic to K as an extension of k.

(b) Prove that $\mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ has just one point if and only if K is a purely inseparable extension of k, that is, an extension obtained by adjoining a sequence of p-th roots, where $\mathrm{char}(k) = p$.

5. If X is a scheme over k and R is a k algebra, we write $X(R)$ for the set $\mathrm{Hom}_k((\mathrm{Spec}\, R), X)$ of R valued points of X. In the case $X = \mathbb{A}^n_k$, we have $X(R) = R^n$, which is not just a set but a an abelian group and a module over the ring $\mathbb{A}^1_k(R) = R$.

(a) Verify that the these structures are functorial in R, e.g., the addition map $R^n \times R^n \to R_n$ is functorial in R, and likewise for other maps describing the zero element, additive inverse, and scalar multiplication by R on R^n.

(b) Construct explicitly the morphism $\mathbb{A}^n_k \times_k \mathbb{A}^n_k \to \mathbb{A}^n_k$ which induces the addition map on $R^n = \mathbb{A}^n_k(R)$, and likewise for the other maps in part (a).

(c) Show that there are commutative diagrams among these morphism which express the axioms of an abelian group and a module. In other words, $\mathbb{A}^n_k$ is an *abelian group scheme* over k, and also an $\mathbb{A}^1_k$ *module scheme* (and $\mathbb{A}^1_k$ is a *commutative ring scheme*).

6. (a) Let $M = \mathbb{A}^{n^2}(k) = \mathrm{Spec}\, k[x_{1,1}, \ldots, x_{n,n}]$. Let X be the open subscheme M_f where $f \in \mathrm{Spec}\, k[x_{1,1}, \ldots, x_{n,n}]$ is the determinant of the $n \times n$ matrix whose with entries $x_{i,j}$. Construct morphisms $\mu\colon X \times_k X \to X$, $i\colon X \to X$ and $e\colon \mathrm{Spec}(k) \to X$ which, for every scheme T over k, equip the set $X(T)_k$ with the product, inverse and identity element of a group, in such a way that for every R algebra k, if $T = \mathrm{Spec}(R)$, then $X(T)_k \cong GL_n(R)$.

(b) Describe the group $X(T)_k$ for general schemes T over k.

7. Using our provisional definition of projective space $\mathbb{P}^n_k$ over a ring k in terms of its covering by standard affine open subsets, prove that $\mathbb{P}^n_k \cong \mathrm{Spec}(k) \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{P}^n_{\mathbb{Z}}$ as a scheme over k (note that every scheme is a scheme over $\mathbb{Z}$ in a unique way). Thus $\mathbb{P}^n_{\mathbb{Z}}$ provides the universal model for projective space from which all others are obtained by base extension.

8. Denoting the group scheme in problem 6 by $(GL_n)_k$, prove that $(GL_n)_k \cong \mathrm{Spec}(k) \times_{\mathrm{Spec}(\mathbb{Z})} (GL_n)_{\mathbb{Z}}$ as a group scheme over k.

9. (a) With the abelian group scheme structure on $\mathbb{A}^n_k$ from problem 5, prove that if k is a field of characteristic $p \neq 0$, and $I \subseteq k[x_1, \ldots, x_n]$ is the ideal generated by the p-th powers x_i^p, then the closed subscheme $\mathrm{Spec}\, k[x_1, \ldots, x_n]/I$ of $\mathbb{A}^n_k$ is a subgroup scheme. This gives examples of non-reduced group schemes of finite type over a field, something which only exists in positive characteristic. See also part (c) of this problem.

(b) Show that $C_n = \mathrm{Spec}\, \mathbb{Z}[x]/(x^n - 1)$ can be identitfied with a subgroup scheme (over $\mathrm{Spec}\, \mathbb{Z}$) of $(GL_1)_{\mathbb{Z}}$, in the notation of Problem 8, and that for any ring R this identifies $C_n(R)$ with the group of n-th roots of 1 in R.

(c) Let k be an algebraically closed field. Show that if $\mathrm{char}(k)$ does not divide n, then the group scheme $\mathrm{Spec}(k) \times_{\mathrm{Spec}(\mathbb{Z})} C_n$ over k is reduced, *i.e.*, it is a classical algebraic variety, whose underlying set can be naturally identified with the group of n-th roots of 1 in k. Show that if $\mathrm{char}(k)$ does divide n, then $\mathrm{Spec}(k) \times_{\mathrm{Spec}(\mathbb{Z})} C_n$ is a non-reduced group scheme over k.

10. Let $G_m = (GL_1)_k = \mathrm{Spec}(k[z, z^{-1}])$, regarded as a group scheme over k as in problems 6 and 8, act on $\mathbb{A}^2_k$ by the formula $(z) \cdot (x, y) = (zx, z^{-1}y)$; in other words, the morphism $\rho\colon G_m \times \mathbb{A}^2 \to \mathbb{A}^2$ which defines the action corresponds to the k algebra homomorphism $\phi\colon k[x, y] \to k[z, z^{-1}] \otimes k[x, y] = k[z, z^{-1}, x, y]$ that maps $x \mapsto zx$, $y \mapsto z^{-1}y$.

(a) Let X be the open subscheme $\mathbb{A}^2 \setminus V(x, y)$. Show that X is G_m invariant, *i.e.*, the action ρ restricted to $G_m \times X$ has image X.

(b) Let Y be the non-separated scheme obtained by gluing two copies of $\mathbb{A}^1$ along the identity map on the open subscheme $U = \mathbb{A}^1 \setminus V(x)$. Construct a surjective morphism $f\colon X \to Y$ which is equivariant for the above G_m action on X and the trivial G_m action on Y.

(c) Show that G_m acts freely on X with quotient $X/G_m = Y$, in the following precise sense: there is a covering of Y by open subschemes V such that $f_V\colon f^{-1}(V) \to V$ makes $f^{-1}(V)$ isomorphic to $G_m \times V$ as a scheme over V with G_m action (where G_m acts on $G_m \times V$ by the left action of G_m on itself and the trivial action on V).

Note that this example is not just an artificial pathology having to do with the definition of schemes. If you like, you can take $k = \mathbb{C}$, so all schemes here are classical algebraic varieties. Then the group variety $\mathbb{C}^*$ acts freely on the open subvariety $X = \mathbb{C}^2 \setminus \{(0, 0)\}$, the orbits are the hyperbolas $xy = c$ for $c \neq 0$, plus the two components of $xy = 0$, and the quotient is genuinely the non-separated variety "$\mathbb{C}^1$ doubled at $\{0\}$."

11. Let $X = Y = \mathbb{C}^*$ and consider the map $f\colon X \to Y$, $z \to z^2$. The two-element group Z_2 acts on X by $z \to -z$, and the action commutes with f (where Z_2 acts trivially on Y).

In the analytic topology, f makes X a principal Z_2 bundle over Y, that is, we can cover Y by open sets U such that $f^{-1}(U) \to U$ is isomorphic to the projection $Z_2 \times U \to U$, as a space over U with Z_2 action.

(a) Show that X, Y and Z_2 can be identified with the sets of closed points of (affine) schemes of finite type over $\mathbb{C}$, so that f is a morphism, and Z_2 is a group scheme which acts on X as a scheme over Y.

(b) Show that for every closed point $y \in Y$, the scheme-theoretic fiber $f^{-1}(y)$ is isomorphic to Z_2 with its usual left action on itself.

(c) Show that X is *not* a principal Z_2 bundle in the Zariski topology, in fact there is no non-empty open subscheme $U \subseteq Y$ such that $f^{-1}(U)$ is isomorphic to $Z_2 \times U$ as a scheme over U. The easiest way to see this is by considering generic points. Note, however, that by the equivalence between X_{cl} and X for Jacobson schemes X, it follows that the result also holds if we omit the generic points. So this is really about the difference between the Zariski topology and the analytic topology on the classical points.

(d) Show that X is a principal Z_2 bundle in the following weaker sense: there exists a surjective morphism $U \to Y$ (U also of finite type over $\mathbb{C}$) such that $U \times_Y X$ is isomorphic to $Z_2 \times U$ as a scheme over U with Z_2 action. Hint: $U = X$ works. In fact, since $U = X \to Y$ is in this case a smooth morphism of relative dimension 0, or *étale* morphism, this shows that X is a 'principal Z_2 bundle in the étale topology.'

12. A *diagram* in a category C is a directed graph $D = (V, E)$ together with a morphism of directed graphs from D to C regarded as a directed graph; that is, a map from $v \mapsto X_v$ from V to objects of C and a map $e \mapsto f_e$ from E to arrows of C such that f_e is an arrow from X_v to X_w if e is an edge from v to w.

A *projective limit* $\varprojlim_D(X_v)$ is an object X with arrows $\alpha_v \colon X \to X_v$ for all $v \in V$ such that $f_e \circ \alpha_v = \alpha_w$ for every edge $e \in E$, where e goes from v to w, and such that for any Y and arrows $\beta_v \colon Y \to X_v$ satisfying the same commutativity condtion, there is a unique arrow $\phi \colon X \to Y$ such that $\beta_v = \alpha_v \circ \phi$ for all v.

As usual for an object defined by a universal property, $\varprojlim_D(X_v)$ is unique up to canonical isomorphism if it exists.

(a) Show that products, fiber products and equalizers are special cases of projective limits of diagrams. (An *equalizer* of two arrows $f, g \colon X \to Y$ is an object E with an arrow $\alpha \colon E \to X$ such that $f \circ \alpha = g \circ \alpha$, and for every $\phi \colon T \to X$ such that $f \circ \phi = g \circ \phi$, ϕ factors through a unique arrow $T \to E$.)

(b) Show that every projective limit $\varprojlim_D(X_v)$ is the equalizer of two morphisms $\prod_v X_v \to \prod_e X_{h(e)}$, where $h(e)$ is the head of e (the vertex e goes to), assuming the two products exist. In particular, a category has all (finite) projective limits if and only if it has all (finite) products and all equalizers.

(c) Show that equalizers are a special case of fiber products.

(d) Deduce that the category of schemes over a given scheme S has all finite projective limits.

13. A morphism f is said to have property P *universally* if every base extension of f has property P. Prove that the property "f has property P universally" is stable under base extension, no matter what P is.

14. Let $f \colon X \to Y$ be a morphism of schemes over S, and $\Gamma_f = (1_X, f) \colon X \to X \times_S Y$ its graph morphism.

(a) Regarding $X \times_S Y$ as a scheme over $Y \times_S Y$ via the morphism $f \times 1_Y$, prove that Γ_f can be identified with the base extension from $Y \times_S Y$ to $X \times_S Y$ of the diagonal morphism $\Delta_{Y/S} \colon Y \to Y \times_S Y$. Use this to prove each of the next three statements.

(b) The graph morphism Γ_f is always an immersion (take as known that the diagonal morphism of any morphism is an immersion).

(c) ('Closed graph theorem') if Y is separated over S, then Γ_f is a closed immersion.

(d) If Y is quasi-separated over S, then Γ_f is quasi-compact.

(e) Prove that in (c) and (d), the hypothesis on Y is necessary for the conclusion to hold for every S morphisms $f \colon X \to Y$. Hint: what is the graph morphisms of the identity map on Y?

Remark: part (c) is an analog of the theorem in topology that if Y is a Hausdorff space, then the graph of every continuous map $f \colon X \to Y$ is closed in $X \times Y$, and part (e) of the fact that the Hausdorff condition on Y is necessary.