Math 256 Homework Set 5

1. Let X be a scheme. By definition (EGA I, 4.1.3), a subscheme of X is a closed subscheme of an open subscheme; and by (EGA I, 4.1.6), any open subscheme of a closed subscheme is a subscheme. However, it is not true in general that every subscheme is open in a closed subscheme.

Prove that if Y is a closed subscheme of a *quasicompact* open $U \subseteq X$, then $Y = U \cap \overline{Y}$, where $\overline{Y}$ is the smallest closed subscheme containing Y, that is, the scheme-theoretic closure of the image of the inclusion morphism $j\colon Y \hookrightarrow X$, defined by the largest quasi-coherent ideal sheaf contained in $\ker(j^\flat)$.

Deduce that if the underlying space of X is locally Noetherian, then every subscheme of Y is open in its closure.

2. (a) Let $R = k[t, x_1, x_2, \ldots]$, where k is a field, and take $X = \operatorname{Spec}(R)$. For each $m > 0$, define an ideal $I_m \subseteq R$ by

$$I_m = (t^m, x_1, \ldots, x_m).$$

Let $U = X \setminus V(t, x_1, x_2 \ldots)$, that is, the complement of the closed point corresponding to the origin in the infinite dimensional affine space X. Define $\mathcal{I} \subseteq \mathcal{O}_U$ to be

$$\mathcal{I} = \bigcap_m (\widetilde{I_m}|U).$$

Prove that $\mathcal{I}$ is quasi-coherent, and that if $Y \subseteq U$ is the closed subscheme defined by $\mathcal{I}$, then $Y \neq U \cap \overline{Y}$. (Note that, by Problem 1, it was necessary to use a non-Noetherian ring R to see this phenomenon.)

(b) Show that the example in part (a) is nevertheless an open subscheme $V \cap \overline{Y}$, for a smaller open subset $V \subseteq U$ which contains Y.

(c) Prove that the union Y' of Y and $V(t-1)$, which is again a closed subscheme of U, is not the intersection of $\overline{Y'}$ with any open subset $V \subseteq X$, and hence is not an open subscheme of a closed subscheme of X.

3. Let X be any topological space, let $P = \{p\}$ be a one-point space, and let $\pi\colon X \to P$ be the unique map. Given any set, abelian group, or ring A, there is a unique sheaf $\underline{A}$ on P such that $\underline{A}_p = A$. Let us also use the notation $\underline{A}$ for the sheaf $\pi^{-1}(\underline{A})$ on X, called the *constant sheaf* with values in A. In particular, $\underline{A}_x = A$ for all $x \in X$.

(a) Prove that $\underline{A}$ is isomorphic to the sheaf of continuous A valued functions on open sets in X, where A is given the discrete topology.

(b) Prove that to give a sheaf homomorphism $\underline{A} \to \mathcal{F}$, where $\mathcal{F}$ is any sheaf on X, it is equivalent to give a homomorphism (of sets, abelian groups, or rings) $A \to \mathcal{F}(X)$.

(c) Show that for any continuous map $f\colon X \to Y$, there is a canonical sheaf homomorphism $\underline{A}_Y \to f_*\underline{A}_X$.

(d) Assume that $A \neq 0$. Prove that if $i\colon Y \to X$ is the inclusion of a disconnected closed subset Y into a connected space X, the canonical map $\underline{A}_Y \to i_* \underline{A}_X$ is an example of a surjective homomorphism of sheaves which induces a non-surjective map on global sections.

(e) Prove that if $\phi\colon M \to N$ is a surjective homomorphism of sheaves which is not surjective on global sections, and $K = \ker(\phi)$, then K is an example of a subsheaf of M such that the presheaf quotient M/K is not a sheaf. In particular, the construction in part (d) provides such examples.

4. If X and T are schemes over S, define $X_S(T) = \mathrm{Hom}_{\mathrm{Sch}\,/S}(T, X)$. For X fixed, this defines a functor $X_S\colon T \mapsto X_S(T)$ from schemes over S to sets called the *functor of points* of X. As in (EGA I, 2.5), if $S = \mathrm{Spec}(k)$ we may also use the term *scheme over* k instead of scheme over S, and write X_k instead of X_S.

Now let $S = \mathrm{Spec}(k)$. Show that if $X = \mathrm{Spec}(k[x_1, \ldots, x_n])$ then $X_k(T) = \mathcal{O}(T)^n$. Here the equal sign means there is a functorial identification between the two sets. If $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/I)$, where I is an ideal, show that $X_k(T)$ is the solution set in $\mathcal{O}(T)^n$ of the equations $f(x_1, \ldots, x_n) = 0$ for all $f(x) \in I$. More generally, prove the analogous facts when $X = \mathrm{Spec}(R)$ for R a polynomial ring in perhaps infinitely many variables, or a quotient of such a ring by an ideal (that is, an arbitrary k-algebra with a chosen set of generators).

5. Let X and Y be schemes over S. Show that $U \mapsto X_S(U)$ for every open $U \subseteq Y$ defines a sheaf of sets on Y. (One refers to this property by saying that the functor X_S from schemes over S to sets is a *sheaf in the Zariski topology.* Since not every functor from schemes over S to sets is a sheaf in the Zariski topology, this gives a nontrivial necessary condition for a functor to be representable as the functor of points of some scheme.)

6. Let k be an algebraically closed field. The set X of $m \times n$ matrices M over k such that $\mathrm{rank}(M) \leq r$ is then an affine algebraic variety in $\mathbb{A}_k^{mn}$, defined by the vanishing of all $(r+1) \times (r+1)$ minors of M. Prove that X is an irreducible variety. Hint: find a surjective morphism from an affine space onto X.

7.* Prove that the $(r+1) \times (r+1)$ minors of M generate the ideal of the variety X in Problem 6, or in other words, that the ideal I generated by the minors is prime.

One method of doing this is as follows. Let $J = \sqrt{I}$ be the full ideal of X, so $X = \mathrm{Spec}(k[x]/J)$, where $k[x]$ denotes the polynomial ring in the mn entries of M. The surjective morphism $\mathbb{A}_k^N \to X$ which you found in Problem 6 corresponds to an injective ring homomorphism $k[x]/J \to k[y]$, where $k[y]$ is a polynomial ring in N variables. Composing this with the canonical homomorphism $k[x]/I \to k[x]/J$ gives a homomorphism $\phi\colon k[x]/I \to k[y]$. To show that $I = J$, we want to show that ϕ is injective. One way to do this is to find a set A of monomials in $k[x]$ such that (i) ϕ maps the monomials in A to linearly independent elements of $k[y]$ (for example, to polynomials with distinct leading terms in some suitable ordering of the monomials in $k[y]$) and (ii) A spans $k[x]/I$ (for example, because every monomial not in A is divisible by the leading term of some generator of I, for a suitable ordering of monomials in $k[x]$).

If you find the general case too hard, try it for $m = 2$, $n = 3$, $r = 1$.

8. Let k be a ring and let $I \subseteq k[x_0, \ldots, x_n]$ be an ideal generated by homogeneous polynomials. Recall that we constructed projective space $\mathbb{P}^n_k$ by gluing affine spaces $U_j = \mathrm{Spec}(R_j) \cong \mathbb{A}^n_k$, where R_j is the polynomial ring in the x_i/x_j for $i = 0, \ldots, n$ (excluding $i = j$).

If f is homogeneous of degree d, then for each j we have $f/x_j^d = f(x_0/x_j, \ldots, x_n/x_j)$, and we can form an ideal $I_j \subseteq R_j$ generated by elements of this form for homogeneous $f \in I$. Prove that the ideal sheaves $\widetilde{I_j} \subseteq \mathcal{O}_{U_j}$ and $\widetilde{I_k} \subseteq \mathcal{O}_{U_k}$ coincide on $U_j \cap U_k$, hence are the restrictions of a single quasi-coherent ideal sheaf $\mathcal{I}$ on $\mathbb{P}^n_k$, defining a closed subscheme $V(\mathcal{I}) \subseteq \mathbb{P}^n_k$ whose intersection with U_j is $\mathrm{Spec}(R_j/I_j)$.

Show that when k is an algebraicallly closed field, so the closed points of $\mathbb{P}^n_k$ are just the points of classical projective space, the set of closed points belonging to $V(\mathcal{I})$ is the solution locus $V(I)$ of the equations defined by homogeneous generators f of I.