Math 256 Homework Set 4

1. Verify in detail the gluing construction (EGA 0, 4.1.7) for ringed spaces, as follows.

(a) Suppose given a collection of sets X_λ, and for every two indices λ, μ a subset $V_{\lambda\mu} \subseteq X_\lambda$ and a bijection $\phi_{\mu\lambda}\colon V_{\lambda\mu} \to V_{\mu\lambda}$, satisfying the gluing conditions: $V_{\lambda\lambda} = X_\lambda$, $\phi_{\lambda\lambda} = 1_{X_\lambda}$, and for every three indices λ, μ, ν, $\phi_{\mu\lambda}(V_{\lambda\mu} \cap V_{\lambda\nu}) = V_{\mu\lambda} \cap V_{\mu\nu}$, and $\phi'_{\mu\lambda} = \phi'_{\mu\nu} \circ \phi'_{\nu\lambda}$, where $\phi'_{\mu\lambda}$, $\phi'_{\nu\lambda}$ are the restrictions of $\phi_{\mu\lambda}$, $\phi_{\nu\lambda}$ to $V_{\lambda\mu} \cap V_{\lambda\nu}$ and $\phi'_{\mu\nu}$ is the restriction of $\phi_{\mu\nu}$ to $V_{\nu\lambda} \cap V_{\nu\mu}$.

Prove that the relation on the disjoint union $\coprod_\lambda X_\lambda$ defined by $x \sim y$ if $x \in X_\lambda$, $y \in X_\mu$ and $\phi_{\mu\lambda}(x) = y$ is an equivalence relation. Let X be the set of equivalence classes. Prove that the canonical maps $\iota_\lambda\colon X_\lambda \to \coprod_\lambda X_\lambda \to X$ are injective, that $\iota_\lambda(V_{\lambda\mu}) = \iota_\mu(V_{\mu\lambda})$, and that under the identification of each X_λ with $\iota_\lambda(X_\lambda) \subseteq X$, $\phi_{\lambda\mu}$ corresponds to the identity map on $\iota_\lambda(V_{\lambda\mu})$.

Note that the gluing conditions are necessary, in the sense that they hold automatically in the case where X_λ are subsets of a set X, $V_{\lambda\mu} = X_\lambda \cap X_\mu$, and $\phi_{\mu\lambda}$ is the identity map on $V_{\lambda\mu} = V_{\mu\lambda}$.

(b) Suppose each X_λ is a topological space, each $V_{\lambda\mu} \subseteq X_\lambda$ is open, and each $\phi_{\mu\lambda}$ is a homeomorphism. Let $U_\lambda = \iota(X_\lambda) \subseteq X$. Prove that in the topologies on U_λ and U_μ induced by their identifications with X_λ, X_μ, the subset $U_\lambda \cap U_\mu$ is open in both U_λ and U_μ, and inherits the same topology as a subspace of each.

Deduce that X has a unique topology such that each U_λ is open and each $\iota_\lambda\colon X_\lambda \to U_\lambda$ is a homeomorphism.

(c) Suppose further that on each X_λ we are given a sheaf $\mathcal{A}_\lambda$ of rings, and for each λ, μ an isomorphism $\phi^\flat_{\mu\lambda}\colon \mathcal{A}_\mu|V_{\mu\lambda} \to (\phi_{\mu\lambda})_*(\mathcal{A}_\lambda|V_{\lambda\mu})$, such that the gluing condition in (a) holds for the ringed space isomorphisms $(\phi_{\mu\lambda}, \phi^\flat_{\mu\lambda})$. Let $\mathcal{F}_\lambda$ be the unique sheaf of rings on U_λ such that $\iota_\lambda^{-1}\mathcal{F}_\lambda = \mathcal{A}_\lambda$ (this makes sense since ι_λ is a homeomorphism of X_λ onto U_λ). Prove that there are isomorphisms of sheaves of rings $\theta_{\mu\lambda}\colon \mathcal{F}_\lambda|(U_\lambda \cap U_\mu) \to \mathcal{F}_\mu|(U_\lambda \cap U_\mu)$ which satisfy the gluing condition for sheaves in (EGA 0, 3.1.3), namely for every three indices λ, μ, ν, the restrictions $\theta'_{\mu\lambda}$, $\theta'_{\mu\nu}$, $\theta'_{\nu\lambda}$ of $\theta_{\mu\lambda}$, $\theta_{\mu\nu}$, and $\theta_{\nu\lambda}$ to $U_\lambda \cap U_\mu \cap U_\nu$ satisfy $\theta'_{\mu\lambda} = \theta'_{\mu\nu} \circ \theta'_{\nu\lambda}$.

(d) Prove (EGA 0, 3.1.3). That is, given a space X, open subsets U_λ which cover X, and sheaves (of sets, abelian groups, or rings) $\mathcal{F}_\lambda$ on U_λ, together with isomorphisms $\theta_{\mu\lambda}$ satisfying the gluing condition, there is a sheaf $\mathcal{F}$ on X and isomorphisms $\eta_\lambda\colon \mathcal{F}_\lambda \to \mathcal{F}|U_\lambda$ such that for all λ, μ we have $\theta_{\mu\lambda} = (\eta'_\mu)^{-1} \circ \eta'_\lambda$, where η'_λ, η'_μ are the restrictions of η_λ, η_μ to $U_\lambda \cap U_\mu$.

Moreover, $\mathcal{F}$ and the η_λ are unique up to an isomorphism commuting with all the η_λ.

2. It follows from what we proved in class about Jacobson schemes and morphisms locally of finite type that if $K \subseteq L$ is a field extension, and L is finitely generated as a K algebra, then L is finite algebraic over K. In this problem we'll show that some of the theory of Jacobson schemes follows, conversely, from this theorem on field extensions.

(a) Deduce from the aforementioned theorem that if X is a scheme locally of finite type over a field K, then the closed points $x \in X$ are precisely the points whose residue field $k(x)$

is finite algebraic over K. For this you will also want to use the easy result that an integral domain finite-dimensional over a field is itself a field.

(b) Deduce from (a) that if $f\colon X \to Y$ is a K-morphism between schemes locally of finite type over K, then f sends closed points to closed points (in fact this holds if we only assume that X is locally of finite type).

(c) Prove that if X is locally of finite type over K, then so is every closed, open, or locally closed subscheme of X.

(d) Deduce from (b) and (c) that every non-empty locally closed subset of X contains a closed point of X, hence that X is Jacobson.

Note in particular that (a) and (d) are sufficient to establish the equivalence between reduced schemes locally of finite type over an algebraically closed field K and classical algebraic varieties over K.

3. If A is a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$, then $\mathfrak{m}/\mathfrak{m}^2$ is a k module, that is, a vector space over k. If X is a scheme and $x \in X$, the *Zariski tangent space* $T_x X$ to X at x is defined to be the dual space $(\mathfrak{m}/\mathfrak{m})^*$, where $A = \mathcal{O}_{X,x}$.

(a) Let k be a field and define $T = \operatorname{Spec}(k[t]/(t^2))$, an affine scheme over k. For any scheme X over k, the set of k-morphisms $T \to X$ is called the set of T-valued points of X in the category of schemes over k, and denoted $X(T)_k$. Prove that $X(T)_k$ is in canonical bijection with data consisting of (i) a k-rational point $x \in X$, that is, a point such that the field extension $k \subseteq k(x)$ induced by the structure morphism $X \to \operatorname{Spec}(k)$ is trivial, and (ii) a vector in $T_x X$. Hint: the underlying space of T has only one point.

(b) Let k be algebraically closed, so that the k-rational points of $\mathbb{A}^n_k = \operatorname{Spec}(k[x_1, \ldots, x_n])$, which are the same as its closed points, are identified with classical points in k^n. Prove that the Zariski tangent space of $\mathbb{A}^n_k$ at every closed point is canonically identified with the vector space k^n.

(c) Part (a) implies that if $f\colon X \to Y$ is a morphism of schemes over k, $x \in X$ is a k-rational point, and $y = f(x)$, then f induces a canonical linear map $df_x\colon T_x X \to T_y Y$, called the differential of f at x. Prove that if X is a closed subvariety of $\mathbb{A}^n_k$ (that is, a reduced closed subscheme $V(I) = \operatorname{Spec}(R/I)$, where $I \subseteq R = k[x_1, \ldots, x_n]$ is a radical ideal), then the differential of the inclusion map $X \to \mathbb{A}^n_k$ at any k-rational (*i.e.*, closed) point of X is injective, and identifies $T_x X$ with the subspace of k^n consisting of vectors v such the directional derivative $\partial_v(f)$ at x vanishes for all $f \in I$ (note that derivatives of polynomials make sense formally over any field).

4. Let k be a commutative ring. Let R be a polynomial ring over k in n^2 variables x_{ij}, $1 \leq i, j \leq n$. Thinking of the x_{ij} as the entries of an $n \times n$ matrix M, let $d = \det(M)$ and let A be the localization $R_d = R[1/d]$, so $\operatorname{Spec}(A)$ is the affine open subset $\mathbb{A}^{n^2}_k \setminus V(d)$. Define $GL_n = \operatorname{Spec}(A)$.

(a) Prove that for any scheme T over k, the set $GL_n(T)_k$ of k-morphisms $T \to GL_n$ is canonically identified with the set of invertible $n \times n$ matrices over $\mathcal{O}_T(T)$.

(b) Prove that there are unique morphisms $m\colon GL_n \times_k GL_n \to GL_n$, $e\colon \operatorname{Spec}(k) \to GL_n$ and $i\colon GL_n \to GL_n$ so that for every k-scheme T, the maps $GL_n(T)_k \times GL_n(T)_k \to GL_n(T)_k$, $\{\text{point}\} \to GL_n(T)_k$, and $GL_n(T)_k \to GL_n(T)_k$ induced by m, e and i give the group law, unit element, and inverse in the group of invertible $n \times n$ matrices over $\mathcal{O}_T(T)$. To do this problem before we define products you can provisionally define $GL_n \times_k GL_n$ to be $\operatorname{Spec}(B)$, where B is the localization of a polynomial ring in two sets of n^2 variables x_{ij} and y_{ij}, in which we invert the determinants of the matrices of x variables and of y variables. Then you should prove that k-morphisms $T \to \operatorname{Spec}(B)$ correspond bijectively to $GL_n(T)_k \times GL_n(T)_k$.

(c) Show that even in the simplest case, when k is a field, so $\operatorname{Spec}(k)$ has just one point, and $n = 1$, so $GL_1 = \operatorname{Spec}(k[x, x^{-1}])$, the morphism m does *not* define a group law on the underlying set of the scheme GL_n.

5. In class we showed, using the gluing construction of $\mathbb{P}^n_k$, that if k is a commutative ring, then to any tuple $(a_0, \ldots, a_n) \in k^n$ such that the a_i generate the unit ideal in k, we can associate a k-morphism $\phi\colon \operatorname{Spec}(k) \to \mathbb{P}^n_k$, or equivalently a k-valued point $\phi \in \mathbb{P}^n(k)$ of $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{Z}}$, and that ϕ depends only on the equivalence class of $(a_0, \ldots, a_n)$ under scalar multiplication by invertible elements of k.

(a) Prove that if k is a field, every $\phi \in \mathbb{P}^n(k)$ is of this form, and this identifies $\mathbb{P}^n(k)$ with the set of points $(a_0\colon \cdots : a_n)$ of classical projective space over k.

(b) Prove that if k is a local ring, every $\phi \in \mathbb{P}^n(k)$ is of this form.

(c) Prove that if $k = \mathbb{Z}$, every $\phi \in \mathbb{P}^1(\mathbb{Z})$ is of this form, and this identifies $\mathbb{P}^1(\mathbb{Z})$ with the set of pairs of relatively prime integers (m, n), up to sign. Show that this set is also canonically identified with $\mathbb{P}^1(\mathbb{Q})$; in fact the unique morphism $\operatorname{Spec}(\mathbb{Q}) \to \operatorname{Spec}(\mathbb{Z})$ induces a bijection $\mathbb{P}^1(\mathbb{Z}) \to \mathbb{P}^1(\mathbb{Q})$.

(d)* Generalize part (c) to all n and any principal ideal domain R in the role of $\mathbb{Z}$ (with its fraction field in the role of $\mathbb{Q}$). Actually it's true when R is a unique factorization domain, but to prove that requires the theory of divisors and a better description of the functor of points of $\mathbb{P}^n$.

6. Prove that $\operatorname{Spec}(A)$ has exactly one point if and only if A is not the zero ring, and every element of A is either invertible or nilpotent.

7. Let X be a disconnected scheme, that is, X is the disjoint union of two non-empty open (and therefore closed) subschemes X_1 and X_2. Prove that the ring $\Gamma(X, \mathcal{O})$ is the Cartesian product $\Gamma(X_1, \mathcal{O}) \times \Gamma(X_2, \mathcal{O})$. Conversely, prove that that if X is a scheme and $\Gamma(X, \mathcal{O})$ is a Cartesian product $A_1 \times A_2$, non-trivial in the sense that neither ring A_i is the zero ring, then X is disconnected.

8. Prove that to give a morphism $X \to \operatorname{Spec}(\mathbb{Z} \times \mathbb{Z})$, where X is a scheme, is equivalent to giving a decomposition $X = X_1 \cup X_2$ of X into two disjoint (possibly empty) open subschemes.

9. Let k be any commutative ring. Prove that the open subset $U = \mathbb{A}^2 \setminus V(x, y)$ in $\mathbb{A}^2 = \operatorname{Spec}(k[x, y])$ is not an affine scheme. Hint: first prove that restriction from $\mathcal{O}(\mathbb{A}^2) = k[x, y]$ to $\mathcal{O}(U)$ is an isomorphism of rings.

10. As in (EGA I, 2.4.1), for any scheme X and point $x \in X$, there is a canonical morphism $f\colon \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$, obtained by composing the inclusion $U \to X$ of an affine open neighborhood $U = \operatorname{Spec}(A)$ of x with the morphism $\operatorname{Spec}(\mathcal{O}_{X,x}) \to U$ induced by the canonical ring homomorphism $A \to A_{\mathfrak{p}_x} = \mathcal{O}_{X,x}$.

If $p \in X$ is a point such that $x \in \overline{\{p\}}$, then every open neighborhood of x contains p. Every element of $\mathcal{O}_{X,x}$ is by definition the germ s_x of a section $s \in \mathcal{O}_X(V)$ for some neighborhood V of x, and two sections $s \in \mathcal{O}_X(V)$, $s' \in \mathcal{O}_X(V')$ have the same germ if they coincide on some smaller neighborhood $W \subseteq V \cap V'$. Since p belongs to every neighborhood of x, the germ s_p depends only on s_x. This gives a canonical ring homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,p}$ (for $x \in \overline{\{p\}}$ in any ringed space X, not just schemes).

(a) Prove that f has the description given by (EGA I, 2.4.2). In more detail, let $Z \subseteq X$ be the subspace consisting of points p such that $x \in \overline{\{p\}}$, and make Z a locally ringed space by defining $\mathcal{O}_Z = i^{-1}\mathcal{O}_X$, where $i\colon Z \hookrightarrow X$ is the inclusion. Then i becomes a morphism of locally ringed spaces with $i^\sharp\colon i^{-1}\mathcal{O}_X \to \mathcal{O}_Z$ the identity map. Since Z is the unique open neighborhood of x in Z, the germ map $\mathcal{O}_Z(Z) \to \mathcal{O}_{Z,x} = \mathcal{O}_{X,x}$ is an isomorphism. Then, since Z is a locally ringed space, the inverse isomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_Z(Z)$ induces a morphism $j\colon Z \to \operatorname{Spec}(\mathcal{O}_{X,x})$. Prove that j is an isomorphism, and that $f = i \circ j^{-1}$. Deduce that f does not depend on the choice of U, is a homeomorphism of $\operatorname{Spec}(\mathcal{O}_{X,x})$ onto Z, and for every $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_{X,x})$, $f^\sharp_{\mathfrak{p}}$ is an isomorphism from $\mathcal{O}_{X,p}$ to $(\mathcal{O}_{X,x})_{\mathfrak{p}}$, where $p = f(\mathfrak{p})$. In fact, the canonical ring homorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,p}$ is the inverse of $f^\sharp_{\mathfrak{p}}$ composed with the localization homomorphism $\mathcal{O}_{X,x} \to (\mathcal{O}_{X,x})_{\mathfrak{p}}$.

(b) Prove (EGA I, 2.4.4), that is, if A is a local ring with maximal ideal $\mathfrak{m}$, then every morphism $g\colon \operatorname{Spec}(A) \to X$, where X is a scheme, factors uniquely through the canonical morphism $\operatorname{Spec}(\mathcal{O}_{X,x}) \to X$, where $x = g(\mathfrak{m})$. Deduce that morphisms $g\colon \operatorname{Spec}(A) \to X$ are in canonical bijection with data consisting of a point $x \in X$ together with a local ring homomorphism $\mathcal{O}_{X,x} \to A$.

(c) Deduce as a corollary (EGA I, 2.4.6) that if k is a field, morphisms $\operatorname{Spec}(k) \to X$ are in canonical bijection with data consisting of a point $x \in X$ and a field extension $k(x) \hookrightarrow k$ (it is also easy to prove this directly, for any locally ringed space X).