Math 256 Homework Set 3

On this homework, assume that all (pre)sheaves take values in the category of sets, abelian groups, or rings. More generally everything will hold for sheaves with values in a category K which has all projective limits and filtered inductive limits.

1. Let $X = \{p\}$ be a space with just one point. Show that the global section functor $\Gamma\colon S \mapsto S(X)$ from sheaves (not presheaves!) on X to sets (or abelian groups, or rings) is an equivalence of categories, that is, for every set A there exists a sheaf S_A such that $S_A(X) = A$, and Γ induces an isomorphism from $\mathrm{Hom}(S_A, S_B)$ in the category of sheaves to $\mathrm{Hom}(A, B)$ in the category of sets (or abelian groups, or rings).

More generally, the same holds for sheaves with values in any target category K which has a terminal object E (that is, every object of K has a unique morphism to E).

2. Given a presheaf F on a space X, define $\widehat{F}$ to be the product of the sheaves $(i_x)_* F_x$ for all $x \in X$, where i_x is the inclusion of $\{x\}$ in X, and we identify the stalk F_x of F with a sheaf on $\{x\}$ as in Problem 1.

(a) More explicitly, show that $\widehat{F}(U) = \prod_{x \in U} F_x$ and describe the restriction maps.

(b) Show that the map $j\colon F \to \widehat{F}$ sending $s \in F(U)$ to the element $(s_x)_{x \in U} \in \widehat{F}(U)$ given by the germs $s_x = \rho_x(s) \in F_x$ is a homomorphism of presheaves.

(c) Show that $\widehat{F}$ is a sheaf. (More generally all direct images and products of sheaves are sheaves.)

(d) Prove that F is a sheaf if and only if (i) the map $j\colon F \to \widehat{F}$ is injective, and (ii) the image $j(F)$ is locally determined, in the sense that for any open U and open covering $U = \bigcup_\alpha U_\alpha$, a section $s \in \widehat{F}(U)$ belongs to $j(F)(U)$ if and only its restriction $s|U_\alpha$ belongs to $j(F)(U_\alpha)$ for all α.

3. Let $\mathcal{B}$ be a base of open sets on X in the weak sense that every open subset of X is a union of sets belonging to $\mathcal{B}$ (in EGA a base of open sets is also assumed closed under finite intersections). Define a presheaf $\mathcal{F}$ on $\mathcal{B}$, and its extension $\mathcal{F}'$ to a presheaf on X, as in EGA (0, 3.2.1). The axiom (F_0) in EGA (0, 3.2.2) only makes sense under the assumption that $\mathcal{B}$ is closed under intersections, but we can replace it with:

> (F_1) For every $U \in \mathcal{B}$, every covering $U = \bigcup_\alpha U_\alpha$ of U by sets $U_\alpha \in \mathcal{B}$, and every covering $U_\alpha \cap U_\beta = \bigcup_\gamma U_{\alpha,\beta,\gamma}$ of $U_\alpha \cap U_\beta$ by sets $U_{\alpha,\beta,\gamma} \in \mathcal{B}$, $\mathcal{F}(U)$ is the projective limit of the system formed by all the $F(U_\alpha)$ and $\mathcal{F}(U_{\alpha,\beta,\gamma})$ and all restriction maps from $\mathcal{F}(U_\alpha)$ or $\mathcal{F}(U_\beta)$ to $\mathcal{F}(U_{\alpha,\beta,\gamma})$.

Prove that $\mathcal{F}'$ is a sheaf if and only if $\mathcal{F}$ satisfies axiom (F_1). Hint: use Problem 2 (d).

4. Describe $X = \mathrm{Spec}(\mathbb{Z})$ in detail: list its points and its open subsets (or its closed subsets), calculate $\mathcal{O}_X(U)$ on each open set U and the stalk $\mathcal{O}_{X,x}$ and residue field k_x at each point x, and specify the restriction, germ and evauation maps. Give a similarly detailed description of the sheaf $\widetilde{M}$, and determine the support $\mathrm{Supp}(\widetilde{M})$ (the set of points x where

$\widetilde{M}_x \neq 0$), for each of the following $\mathbb{Z}$ modules (that is, abelian groups): (a) $M = \mathbb{Z}/n\mathbb{Z}$, for any integer $n \notin \{0, \pm 1\}$; (b) $M = \mathbb{Q}$.

5. Repeat Problem 4 with $\mathbb{Z}$ replaced by $k[x]$, where k is a field, and in the last part considering (a) $M = k[x]/f(x)$, for any non-constant, non-zero polynomial $f(x)$; (b) $M = k(x)$. What simplifications occur if k is algebraically closed?

6. Let $I \subseteq R$ be an ideal and M a finitely generated R module. Prove the special case in (EGA 0, 1.7.5): the support $\operatorname{Supp}(M/IM)^{\sim}$ of the sheaf on $\operatorname{Spec}(R)$ associated to the module M/IM is equal to $V(I + \operatorname{ann} M)$ (or, if you prefer, prove the full statement about $\operatorname{Supp}(M \otimes_R N)$ in (EGA 0, 1.7.5) and deduce the special case). Give an example to show that the assumption that M is finitely generated is necessary.

7. Let X be a topological space on which a group G acts by continuous automorphisms. We say that G *acts equivariantly* on a sheaf $\mathcal{F}$ on X if we are given sheaf homomorphisms $\sigma_g \colon \mathcal{F} \to (g^{-1})_* \mathcal{F}$ for all $g \in G$ (so σ_g maps $\mathcal{F}(U)$ to $\mathcal{F}(g(U))$), such that σ_1 is the identity and $\sigma_{gh} = (h^{-1})_*(\sigma_g) \circ \sigma_h$ for all $g, h \in G$. In other words, the G action on X extends to an action on $(X, \mathcal{F})$ in the category of pairs consisting of a space and a sheaf on it. If G acts on another space Y, and $f \colon X \to Y$ is a continuous map commuting with the G actions, show that G acts equivariantly on $f_* \mathcal{F}$ in a natural way.

8. Now for an interesting (I hope) application of the previous problem. Let $X = Y = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, with the usual Hausdorff topology, and let $G = \{1, \tau\}$ be the two-element group acting on X by $\tau(z) = -z$. Let G act trivially on Y. Let $f \colon X = \mathbb{C}^\times \to \mathbb{C}^\times = Y$ be given by $f(z) = z^2$. Let $\mathbb{Z}$ denote the constant sheaf on X such that $\mathbb{Z}(U) = \mathbb{Z}$ for every connected U.

(a) Show that there is a unique equivariant action of G on $\mathbb{Z}$ such that under the identification of $\mathbb{Z}(X)$ with $\mathbb{Z}$, σ_τ acts on $\mathbb{Z}(X)$ as multiplication by -1.

(b) Since G acts trivially on Y, Problem 7 gives an action of G by sheaf automorphisms on $f_*\mathbb{Z}$. Let $\mathcal{F} \subseteq f_*\mathbb{Z}$ be the subsheaf of G-invariant sections of $f_*\mathbb{Z}$. Show that $\mathcal{F}$ is a locally constant sheaf with stalks $\mathbb{Z}$, but not a constant sheaf; in fact $\mathcal{F}(Y) = \{0\}$.

(c) Fix a point $z \in Y$, and let $\mathcal{A}$ be the skyscraper sheaf with stalk $\mathbb{Z}$ at z. Verify that there is a surjective sheaf homomorphism $\mathcal{F} \to \mathcal{A}$ that sends s to its germ s_z, or to zero if $s \in \mathcal{F}(U)$ for $z \notin U$. This is an example of a surjective sheaf homorphism which is not surjective on sections, since $\mathcal{A}(Y) = \mathbb{Z}$.

(d) Let $\mathcal{F}$ be a locally constant sheaf on any space X, let z be a point of X, and let $p \colon [0,1] \to X$, $p(0) = p(1) = z$ represent an element of the fundamental group $\pi_1(X, z)$. Then $p^{-1}\mathcal{F}$ is a *constant* sheaf, so there are canonical identifications $\mathcal{F}_z \cong (p^{-1}\mathcal{F})_0 \cong (p^{-1}\mathcal{F})([0,1]) \cong (p^{-1}\mathcal{F})_0 \cong \mathcal{F}_z$. Composing these gives an an automorphism σ_p of $\mathcal{F}_z$, and this is an action of $\pi_1(X, z)$ on $\mathcal{F}_z$, called the *monodromy action*. Show that in the example above, the locally constant sheaf $\mathcal{F}$ on $Y = \mathbb{C}^\times$ has non-trivial monodromy; in particular a generator of the fundamental group acts as -1 on $\mathcal{F}_z$.

(e) If you used the Zariski topology on $\mathbb{C}^\times$ instead, you would get $\mathcal{F} = 0$. Indeed, since $\mathbb{C}^\times$ is irreducible, any locally constant sheaf is constant in the Zariski topology. So, even though we constructed the covering $f\colon X \to Y$ purely algebraically, there is no locally constant sheaf in the Zariski topology on Y that detects its monodromy.

It is possible to define a purely algebraic notion of "sheaf in the étale topology," which for complex algebraic varieties more closely resembles the notion of sheaf in the classical topology, but also makes sense, for example, in characteristic p. Time permitting, we might discuss this later in the course.

9. Let $A = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$ be a polynomial ring in infinitely many variables over a field k, modulo the ideal generated by the squares of the variables. Show that A has a unique prime ideal $\mathfrak{m} = (x_1, x_2, \ldots)$, but A is not Noetherian. In particular, $\operatorname{Spec}(A)$ being a Noetherian space does not imply that A is a Noetherian ring.

10. Let $A = k[x, y]$, where k is an algebraically closed field.

(a) Show that if two polynomials $p(x, y)$, $q(x, y)$ have no common factor, then the solution set $V(p, q) \subseteq k^2$ is finite.

(b) Deduce that every prime ideal of A is one of the following: (i) the zero ideal, (ii) a maximal ideal $(x - a, y - b)$ for some $(a, b) \in k^2$, or (iii) a principal ideal (f) generated by an irreducible polynomial $f(x, y)$. For (ii), assume Hilbert's Nullstellensatz: if an ideal $I \subseteq A$ is not the unit ideal, then $V(I) \subseteq k^2$ is non-empty.

Note that (b) shows that $\operatorname{Spec}(A)$ consists of closed points in bijection with k^2, together with a generic point for each irreducible algebraic curve in k^2, and a generic point for the whole plane k^2.

11. With A as in Problem 10, let $f(x, y) \in A$ be an irreducible polynomial and $B = k[x, y]/(f)$.

(a) Show that the points of $\operatorname{Spec}(B)$ are (i) the generic point $\mathfrak{p} = (0)$, and (ii) closed points corresponding to points (a, b) on the curve $f(x, y) = 0$ in k^2.

(b) Show that the closed sets in the the Zariski topology on $X = \operatorname{Spec}(B)$ are X itself, and finite subsets of the closed points. In particular, the topology of $\operatorname{Spec}(B)$ does not depend on f, up to homeomorphism, although in general $\operatorname{Spec} k[x, y]/(f)$ and $\operatorname{Spec} k[x, y]/(g)$ need not be isomorphic as schemes.

12. Consider the case of Problem 11 where $f(x, y)$ has the form $y - g(x)$, so the curve is the graph of a function $g(x)$. Show that in this case, $\operatorname{Spec}(B)$ is isomorphic to the affine line $\operatorname{Spec}(k[x])$.