Math 256 Homework Set 2

On this homework set k always stands for an algebraically closed field.

1. Prove that every regular function defined on the whole of projective space $\mathbb{P}^n(k)$ is constant. For this you may assume the theorem, which we will prove later, that every global regular function on an affine variety $V(I) \subseteq k^n$ is given by a polynomial in the coordinates. (Recall that by definition, f is regular if it is given locally by rational functions in the coordinates, so this theorem is not obvious.)

2. Let X be the hypersurface $X = V(xy - wz)$ in $\mathbb{A}^4(k)$. Let $Z \subseteq X$ be the closed subvariety $Z = V(y, z)$ and let $U = X \setminus Z$. Find a regular function $g \in \mathcal{O}_X(U)$ which cannot be expressed in the form $g = h/f$, where h and f are polynomials in the coordinates w, x, y, z such that $f \neq 0$ on U.

3. Prove that the open subvariety U in problem 3 is isomorphic to the complement of a line in $\mathbb{A}^3$, and hence that U is not an affine variety. For the last part you might want to use the example we did in class, that the complement of the origin in $\mathbb{A}^2$ is not affine.

4. Let $(x_0 : \cdots : x_3)$ be projective coordinates on $\mathbb{P}^3$. Let $U_0, \ldots, U_3$ be the standard affines $U_i = \mathbb{P}^3 \setminus V(x_i)$, with coordinates $\{x_j \mid j \neq i\}$ on U_i given by fixing $x_i = 1$.

(a) Show that there is a projective variety $Z \subseteq \mathbb{P}^3$ such that $Z \cap U_i$ is given by the equations

$$\begin{aligned} x_2 = x_1^2,\ x_3 = x_1^3 &\quad \text{on } U_0 \\ x_0x_2 = 1,\ x_3 = x_2^2 &\quad \text{on } U_1 \\ x_1x_3 = 1,\ x_0 = x_1^2 &\quad \text{on } U_2 \\ x_1 = x_2^2,\ x_0 = x_2^3 &\quad \text{on } U_3. \end{aligned}$$

(b) Find homogeneous equations of Z in projective coordinates.

(c) Construct a morphism $\mathbb{P}^1 \to \mathbb{P}^3$ whose image is Z. Is Z isomorphic to $\mathbb{P}^1$?

5. Prove that for every linear polynomial $f(x_0, \ldots, x_n)$, the open subvariety $U = \mathbb{P}^n(k) \setminus V(f)$ is affine.

6.* Prove that for every homogeneous polynomial $f(x_0, \ldots, x_n)$, the open subvariety $U = \mathbb{P}^n(k) \setminus V(f)$ is affine. Hint: for coordinates on U take all the functions x^m/f, where x^m is a monomial in the x_i of degree $d = \deg(f)$. Then show that U is isomorphic to the affine variety X with coordinate ring $\mathcal{O}(X) = R/(f(x) - 1)$ where $R \subseteq k[x_0, \ldots, x_n]$ is the subalgebra generated by monomials of degree d.

7. Given a polynomial $f(x_1, \ldots, x_n)$ over k, we can identify the graph of f with the affine variety $X = V(y - f(x)) \subseteq \mathbb{A}^{n+1}(k)$. Prove that X is isomorphic to $\mathbb{A}^n(k)$, and that $y - f(x)$ generates the ideal $\mathcal{I}(X) \subseteq k[x_1, \ldots, x_n, y]$.

8. Let $X = V(y - f(x)) \subseteq \mathbb{A}^{n+1}(k)$ be the graph of f, as in Problem 7. Construct a natural bijective correspondence between ring homomorphisms $\mathcal{O}(X) \to k[t]/(t^2)$ and pairs (p, v), where $p \in X$ and v is a tangent vector to X at p, that is, a vector in k^{n+1} such that the

directional derivative of $y - f(x)$ in the v direction vanishes at p. Note that the derivatives of a polynomial make sense formally with coefficients in any field.