Math 256 Homework Set 1

1. In class we discussed the parametrization of the curve $C = V(y^2 - x^3)$ via the morphism $\phi\colon \mathbb{A}^1 \to C$ sending the point (t) to (t^2, t^3), and showed that ϕ is bijective but not an isomorphism. Now prove that the subring $k[t^2, t^3]$ of $k[t]$ is not isomorphic to a polynomial ring over k, and explain how this shows that C is not isomorphic to $\mathbb{A}^1$ as an affine variety (by any morphism).

2. Let k be a field of characteristic not equal to 2. Show that every line of the form $V(y - t(x+1))$ in k^2, where $t \in k$ and $t^2 \neq -1$, meets the conic $V(x^2 + y^2 - 1)$ in exactly two points: $(-1, 0)$ and another point $(a(t), b(t))$. Solve for $a(t)$ and $b(t)$, and show that this gives a parametrization of the solutions of $x^2 + y^2 = 1$ in k^2, omitting $(-1, 0)$, by the open subset $k \setminus \{\pm\sqrt{-1}\}$ of k, even if k is not assumed to be algebraically closed. Deduce that every Pythagorean triple of coprime positive integers $a^2 + b^2 = c^2$ has the form $\{2pq, (p+q)(p-q), p^2+q^2\}$ for some integers $p > q > 0$.

3. Extend the parametrization in Problem 2 to a morphism from $\mathbb{P}^1$ to the projective closure C of $V(x^2 + y^2 - 1)$, and show that it is an isomorphism. Assuming k algebraically closed, what 3 points on C correspond to the points $\pm\sqrt{-1}$ and ∞ on $\mathbb{P}^1$ which one omits to get the morphism with domain $\mathbb{A}^1(k) \setminus \{\pm\sqrt{-1}\}$ in Problem 2?

4. (a) Show that $\mathbb{P}^9$ parametrizes the set of all cubic curves in $\mathbb{P}^2$, and find the equation of the universal family $H \subseteq \mathbb{P}^9 \times \mathbb{P}^2$ whose fiber over a point of $\mathbb{P}^9$ is the curve parametrized by that point.

(b*) Let Y be the locus in $\mathbb{P}^9$ which parametrizes cubic curves that decompose into a line and a conic (including as degenerate cases 3 lines, a line and a double line, or a triple line). Show that Y is a closed subvariety of $\mathbb{P}^9$ and find its equation(s).

5. We can regard $\mathrm{SL}_2(k)$ as the affine variety $V(ad - bc - 1)$ in the space $\mathbb{A}^4(k)$ of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that the group law in $\mathrm{SL}_2(k)$ and the map sending an element to its inverse are morphisms of algebraic varieties.

6. The group $\mathrm{SL}_2(k)$ acts on the vector space k^2, and therefore on $\mathbb{P}^1(k)$, which we can identify with the set of one-dimensional subspaces of k^2: explicitly, the action sends $(z : w) \in \mathbb{P}^1$ to $(z' : w')$, where

$$\begin{pmatrix} z' \\ w' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.$$

(a) Show that on the affine line $\{(z : 1)\} \subset \mathbb{P}^1$, this action is the fractional linear transformation sending (z) to $(az+b)/(cz+d)$, for $cz + d = 0$, extending to all of $\mathbb{P}^1$ by sending $(-d : c)$ to $(1 : 0)$ and $(1 : 0)$ to $(a : c)$.

(b) Show that the action of any fixed matrix in $\mathrm{SL}_2(k)$ is a morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$, by covering $\mathbb{P}^1$ with affine open sets on which it is given by a polynomial map (you will need different open coverings on $\mathbb{P}^1$ regarded as the domain and codomain of the map).

(c) Show that the action of the whole group is a morphism $\mathrm{SL}_2(k) \times \mathbb{P}^1 \to \mathbb{P}^1$.