1. In class we discussed the parametrization of the curve \( C = V(y^2 - x^3) \) via the morphism \( \phi: \mathbb{A}^1 \rightarrow C \) sending the point \((t)\) to \((t^2, t^3)\), and showed that \( \phi \) is bijective but not an isomorphism. Now prove that the subring \( k[t^2, t^3] \) of \( k[t] \) is not isomorphic to a polynomial ring over \( k \), and explain how this shows that \( C \) is not isomorphic to \( \mathbb{A}^1 \) as an affine variety (by any morphism).

2. Let \( k \) be a field of characteristic not equal to 2. Show that every line of the form \( V(y - t(x + 1)) \) in \( k^2 \), where \( t \in k \) and \( t^2 \neq -1 \), meets the conic \( V(x^2 + y^2 - 1) \) in exactly two points: \((-1, 0)\) and another point \((a(t), b(t))\). Solve for \( a(t) \) and \( b(t) \), and show that this gives a parametrization of the solutions of \( x^2 + y^2 = 1 \) in \( k^2 \), omitting \((-1, 0)\), by the open subset \( k \setminus \{ \pm \sqrt{-1} \} \) of \( k \), even if \( k \) is not assumed to be algebraically closed. Deduce that every Pythagorean triple of coprime positive integers \( a^2 + b^2 = c^2 \) has the form \( \{2pq, (p + q)(p - q), p^2 + q^2 \} \) for some integers \( p > q > 0 \).

3. Extend the parametrization in Problem 2 to a morphism from \( \mathbb{P}^1 \) to the projective closure \( C = V(x^2 + y^2 - 1) \), and show that it is an isomorphism. Assuming \( k \) algebraically closed, what 3 points on \( C \) correspond to the points \( \pm \sqrt{-1} \) and \( \infty \) on \( \mathbb{P}^1 \) which one omits to get the morphism with domain \( \mathbb{A}^1(k) \setminus \{ \pm \sqrt{-1} \} \) in Problem 2?

4. (a) Show that \( \mathbb{P}^3 \) parametrizes the set of all cubic curves in \( \mathbb{P}^2 \), and find the equation of the universal family \( H \subseteq \mathbb{P}^3 \times \mathbb{P}^2 \) whose fiber over a point of \( \mathbb{P}^3 \) is the curve parametrized by that point.

(b*) Let \( Y \) be the locus in \( \mathbb{P}^3 \) which parametrizes cubic curves that decompose into a line and a conic (including as degenerate cases 3 lines, a line and a double line, or a triple line). Show that \( Y \) is a closed subvariety of \( \mathbb{P}^3 \) and find its equation(s).

5. We can regard \( \text{SL}_2(k) \) as the affine variety \( V(ad - bc - 1) \) in the space \( \mathbb{A}^4(k) \) of \( 2 \times 2 \) matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Show that the group law in \( \text{SL}_2(k) \) and the map sending an element to its inverse are morphisms of algebraic varieties.

6. The group \( \text{SL}_2(k) \) acts on the vector space \( k^2 \), and therefore on \( \mathbb{P}^1(k) \), which we can identify with the set of one-dimensional subspaces of \( k^2 \): explicitly, the action sends \((z : w) \in \mathbb{P}^1\) to \((z' : w')\), where

\[
\begin{pmatrix} z' \\ w' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.
\]

(a) Show that on the affine line \( \{(z : 1)\} \subseteq \mathbb{P}^1 \), this action is the fractional linear transformation sending \((z)\) to \((az + b)/(cz + d)\), for \( cz + d = 0 \), extending to all of \( \mathbb{P}^1 \) by sending \((-d : c)\) to \((1 : 0)\) and \((1 : 0)\) to \((a : c)\).

(b) Show that the action of any fixed matrix in \( \text{SL}_2(k) \) is a morphism from \( \mathbb{P}^1 \) to \( \mathbb{P}^1 \), by covering \( \mathbb{P}^1 \) with affine open sets on which it is given by a polynomial map (you will need different open coverings on \( \mathbb{P}^1 \) regarded as the domain and codomain of the map).
(c) Show that the action of the whole group is a morphism $\text{SL}_2(k) \times \mathbb{P}^1 \to \mathbb{P}^1$. 