NOTES ON SHEAF COHOMOLOGY FOR SCHEMES

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1. Čech resolutions

Let M be a sheaf of abelian groups on a topological space X. Given $U \subseteq X$ open, we write $j_U \colon U \to X$ for the inclusion. Then we have a canonical functorial sheaf homomorphism $M \to (j_U)_* \, j_U^{-1} M$, the co-unit of the adjunction between the functors j_U^{-1} and $(j_U)_*$. More explicitly, $j_U^{-1} M = M|U$, so for any open $V \subseteq X$ we have $((j_U)_* \, j_U^{-1} M)(V) = (M|U)(U \cap V) = M(U \cap V)$, and $M(V) \to ((j_U)_* \, j_U^{-1} M)(V)$ is just restriction $\rho^V_{U\cap V}$.

The canonical homomorphism $M \to (j_U)_* \, j_U^{-1} M$ restricts to an isomorphism on U and hence on any $V \subseteq U$. Let $j_V^{-1}(j_U)_* \, j_U^{-1} M \to j_V^{-1} M$ be its inverse on V. By the adjunction between j_V^{-1} and $(j_V)_*$, this corresponds to a homomorphism $(j_U)_* \, j_U^{-1} M \to (j_V)_* \, j_V^{-1} M$. Put another way, the co-unit $1 \to (j_V)_* \, j_V^{-1}$ applied to the sheaf $(j_U)_* \, j_U^{-1} M$ gives a canonical homomorphism $(j_U)_* \, j_U^{-1} M \to (j_V)_* \, j_V^{-1}(j_U)_* \, j_U^{-1} M$, and we can identify $j_V^{-1}(j_U)_* \, j_U^{-1} M$ with $j_V^{-1} M$ to get $(j_U)_* \, j_U^{-1} M \to (j_V)_* \, j_V^{-1} M$. In terms of sections we have $((j_U)_* \, j_U^{-1} M)(W) = M(U \cap W)$, $((j_V)_* \, j_V^{-1} M)(W) = M(V \cap W)$, and the homomorphism $((j_U)_* \, j_U^{-1} M)(W) \to ((j_V)_* \, j_V^{-1}(j_U)_* \, j_U^{-1} M)(W)$ is given by restriction $\rho^{U\cap W}_{V\cap W}$.

In particular, for open sets $U'' \subseteq U' \subseteq U$ we see that the composite of canonical maps $(j_U)_* \, j_U^{-1} M \to (j_{U'})_* \, j_{U'}^{-1} M \to (j_{U''})_* \, j_{U''}^{-1} M$ is equal to the canonical map $(j_U)_* \, j_U^{-1} M \to (j_{U''})_* \, j_{U''}^{-1} M$.

Lemma 1.1. *For any two open sets U and V there is a canonical identification*

$$(j_{U\cap V})_* \, j_{U\cap V}^{-1} M = (j_U)_* \, j_U^{-1}(j_V)_* \, j_V^{-1} M \tag{1}$$

compatible with the canonical homomorphisms described above.

Proof. On any open set W the set of sections of the sheaf on either side in (1) is $M(U \cap V \cap W)$, so the two sheaves are equal. Even without being precise about what 'compatible' means, it is clear from this description that all reasonable compatibilities must hold. $\square$

Now let $\mathcal{F} = (U_\alpha)_{\alpha\in\mathcal{I}}$ be a collection of open subsets of X. For each finite subset $I \subseteq \mathcal{I}$ of the index set, we put $U_I = \bigcap_{\alpha\in I} U_\alpha$; in particular, $U_\emptyset = X$. We build a complex of sheaves $\mathcal{C}^\bullet(M, \mathcal{F})$, zero in negative degrees, with

$$\mathcal{C}^n(M, \mathcal{F}) = \prod_{I\subseteq\mathcal{I},\, |I|=n} (j_{U_I})_* \, j_{U_I}^{-1} M \tag{2}$$

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for $n \geq 0$. The differentials in $\mathcal{C}^\bullet(M, \mathcal{F})$ are constructed as follows. For each $I \subseteq J \subseteq \mathcal{I}$, with $|I| = n$ and $|J| = n+1$, we have $U_J \subseteq U_I$, and we get a map $p_{I,J} \colon \mathcal{C}^n(M, \mathcal{F}) \to (j_{U_J})_* \, j_{U_J}^{-1} M$ by composing the canonical map $(j_{U_I})_* \, j_{U_I}^{-1} M \to (j_{U_J})_* \, j_{U_J}^{-1} M$ with the projection $\mathcal{C}^n(M, \mathcal{F}) \to (j_{U_I})_* \, j_{U_I}^{-1} M$ on a component of the product in (2). To determine signs, we fix a total ordering of the index set $\mathcal{I}$. For each $J \subseteq \mathcal{I}$ of size $n+1$, let $J = \{\alpha_0, \ldots, \alpha_n\}$ with the elements listed in order, and define $q_J \colon \mathcal{C}^n(M, \mathcal{F}) \to (j_{U_J})_* \, j_{U_J}^{-1} M$ by $q_J = \sum_i (-1)^i p_{J \setminus \{\alpha_i\}, J}$. Finally we define $d^n \colon \mathcal{C}^n(M, \mathcal{F}) \to \mathcal{C}^{n+1}(M, \mathcal{F})$ by taking its projection on the component $(j_{U_J})_* \, j_{U_J}^{-1} M$ of $\mathcal{C}^{n+1}(M, \mathcal{F})$ to be q_J. It is an easy exercise to see that the sign rule in the definition of q_J makes $d^{n+1} \circ d^n = 0$.

Definition 1.2. The complex $\mathcal{C}^\bullet(M, \mathcal{F})$, $d^\bullet$ is the *Čech complex* of M with respect to the (ordered) collection $\mathcal{F}$.

Up to isomorphism, the Čech complex does not depend on the chosen ordering of the index set $\mathcal{I}$. More precisely, given two different orderings, there is an isomorphism between the corresponding versions of the Čech complex, given on each term $\mathcal{C}^n(M, \mathcal{F})$ by an automorphism which is multiplication by ± 1 on each component.

Lemma 1.3. *Fix an index $\alpha \in \mathcal{I}$ and define $\mathcal{I}' = \mathcal{I} \setminus \{\alpha\}$ and $\mathcal{F}' = (U_\beta)_{\beta \in \mathcal{I}'}$ Then $\mathcal{C}^\bullet(M, \mathcal{F})$ is isomorphic to the -1 shift of the mapping cone of the canonical homomorphism of complexes of sheaves*

$$\mathcal{C}^\bullet(M, \mathcal{F}') \to (j_{U_\alpha})_* \, j_{U_\alpha}^{-1} \mathcal{C}^\bullet(M, \mathcal{F}'). \tag{3}$$

Proof. The essential point is the identification $(j_{U_\alpha})_* \, j_{U_\alpha}^{-1} (j_V)_* \, j_V^{-1} M = (j_{U_\alpha \cap V})_* \, j_{U_\alpha \cap V}^{-1} M$ for any open set V given by Lemma 1.1. Using this, we can decompose $\mathcal{C}^n(M, \mathcal{F})$ as $\mathcal{C}^n(M, \mathcal{F}') \oplus (j_{U_\alpha})_* \, j_{U_\alpha}^{-1} \mathcal{C}^{n-1}(M, \mathcal{F}')$, in which the first term is the product of the factors $(j_{U_I})_* \, j_{U_I}^{-1} M$ for $\alpha \notin I$ and the second term is identified with the product of the factors for $\alpha \in I$. Here we are also using the fact that the functors $(j_{U_\alpha})_*$ and $j_{U_\alpha}^{-1}$ commute with products: $(j_{U_\alpha})_*$ because any direct image commutes with products, and $j_{U_\alpha}^{-1}$ because it is same as the restriction functor $M \mapsto M|U_\alpha$.

This decomposition identifies each term $\mathcal{C}^n(M, \mathcal{F})$ with the $(n-1)$-st term of the cone on the map in (3). Matching up the differentials is a simple exercise in sign bookkeeping. □

For $\mathcal{I}$ finite, Lemma 1.3 implies that we could construct $\mathcal{C}^\bullet(M, \mathcal{F})$ (up to shift) as an iterated mapping cone. However, we do not want to take that as a definition since it only applies in the finite case.

By definition, $j_{U_\emptyset}$ is the identity map 1_X, so the 0-th term of the Čech complex is just $\mathcal{C}^0(M, \mathcal{F}) = M$. If $\mathcal{F}$ is a covering of X, that is, if $X = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$, then, according to the next result, the rest of the complex is a resolution of M.

Corollary 1.4. *If $\mathcal{F}$ is a covering of X, then the complex*

$$\mathcal{C}^\bullet(M, \mathcal{F}) = (0 \to M \to \mathcal{C}^1(M, \mathcal{F}) \to \cdots) \tag{4}$$

is acyclic.

Proof. More generally we claim that for any $\mathcal{F}$, the Čech complex is acyclic on $V = \bigcup_{\alpha \in I} U_\alpha$. Equivalently, the complex of stalks at x is acyclic for all $x \in V$. Choosing some α such that $x \in U_\alpha$, the homomorphism in (3) is an isomorphism on U_α and in particular on stalks at x. So the complex of stalks is the mapping cone of an isomorphism, hence acyclic. $\square$

Remark 1.5. Let $V = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$ and let $\widehat{\mathcal{C}}^\bullet(M, \mathcal{F})$ denote the shifted Čech complex $\mathcal{C}^\bullet(M, \mathcal{F})[1]$ with its degree -1 term $\mathcal{C}^0(M, \mathcal{F}) = M$ deleted. The proof of Corollary 1.4 shows that $M|V$ is quasi-isomorphic to $\widehat{\mathcal{C}}^\bullet(M, \mathcal{F})|V$. Since $\widehat{\mathcal{C}}^\bullet(M, \mathcal{F})|V$ is zero in negative degrees, and Γ_V is a left exact functor, we have $H^0\Gamma_V\widehat{\mathcal{C}}^\bullet(M, \mathcal{F}) = M(V)$. Interpreting the left hand side in terms of sections of M on the covering $\mathcal{F}$ of V, this identity simply expresses the fact that M satisfies the sheaf axiom.

If M has additional structure compatible with direct images, products, and restriction to open sets, then so does the Čech complex $\mathcal{C}^\bullet(M, \mathcal{F})$. In particular, if X is equipped with a sheaf of rings $\mathcal{O}_X$ and M is a sheaf of $\mathcal{O}_X$ modules, then $\mathcal{C}^\bullet(M, \mathcal{F})$ is a complex of sheaves of $\mathcal{O}_X$ modules. From now on we will assume that M is a sheaf of modules for some specified sheaf of rings on X. The Čech complex of a sheaf of abelian groups is the special case of this in which the sheaf of rings is the constant sheaf $\underline{\mathbb{Z}}$ with stalks $\mathbb{Z}$.

2. Čech acyclic sheaves

Let $\mathcal{B}$ be a base of open neighborhoods on X which is closed under finite intersections; in particular, we require $X \in \mathcal{B}$ as the 'void intersection.'

Definition 2.1. A sheaf of $\mathcal{O}_X$ modules M on X is $\mathcal{B}$-*acyclic* if the complex $\Gamma_U\mathcal{C}^\bullet(M, \mathcal{F})$ is acyclic for every $U \in \mathcal{B}$ and every covering $\mathcal{F}$ of U by subsets $U \supseteq U_\alpha \in \mathcal{B}$.

Lemma 2.2. *Flasque sheave are $\mathcal{B}$-acyclic.*

Proof. Let $\mathcal{F}$ be a cover in $\mathcal{B}$ of $U \in \mathcal{B}$. Since flasque sheaves are preserved by restriction to open sets, direct images, and products, $\mathcal{C}^\bullet(M, \mathcal{F})|U$ is a complex of flasque sheaves on U, and it is acyclic by Corollary 1.4. Since flasque resolutions compute $R\Gamma$, it follows that $\Gamma_U\mathcal{C}^\bullet(M, \mathcal{F})$ is acyclic. $\square$

Corollary 2.3. *Every sheaf of $\mathcal{O}_X$ modules is a subsheaf of a $\mathcal{B}$-acyclic sheaf.*

In Remark 1.5 we noticed that for $\bigcup_{\mathcal{F}} = V$, we have $H^0\Gamma_V\widehat{\mathcal{C}}(A, \mathcal{F}) = A(V)$ and that this is really just a formulation of the sheaf axiom. The argument we will use in the proof of the next lemma shows that $H^1\Gamma_V\widehat{\mathcal{C}}(A, \mathcal{F})$ also has a fundamental sheaf-theoretic interpretation: it contains the 'obstruction' to lifting a section of a quotient sheaf M/A to a section of M on V.

Lemma 2.4. *If* $0 \to A \to M \to N \to 0$ *is an exact sequence of sheaves and A is $\mathcal{B}$-acyclic, then* $0 \to A(U) \to M(U) \to N(U) \to 0$ *is exact for all $U \in \mathcal{B}$. In particular, this holds for $U = X$.*

Proof. Since Γ_U is left exact we are to show that $M(U) \to N(U)$ is surjective. Let $s \in N(U)$. We need to lift s to a section $\overline{s} \in M(U)$. By the definition of epimorphism of sheaves, this can be done locally, that is, we can cover U by open subsets U_α and find lifts $\overline{s}_\alpha \in M(U_\alpha)$ of $s|U_\alpha$. Since $\mathcal{B}$ is a base of the topology on X, we can do this with each $U_\alpha \in \mathcal{B}$. Let $\mathcal{F}$ be the collection of these U_α. For all α and β, we have $(\overline{s}_\alpha - \overline{s}_\beta)|(U_\alpha \cap U_\beta) \in A(U_\alpha \cap U_\beta) = \Gamma_U (j_{U_\alpha \cap U_\beta})_* \, j^{-1}_{U_\alpha \cap U_\beta} A$, since both $\overline{s}_\alpha|(U_\alpha \cap U_\beta)$ and $\overline{s}_\beta|(U_\alpha \cap U_\beta)$ are lifts of $s|(U_\alpha \cap U_\beta) \in N(U_\alpha \cap U_\beta)$. This system of sections of A on the various $U_\alpha \cap U_\beta$ defines an element of $\Gamma_U \mathcal{C}^2(A, \mathcal{F})$. Furthermore, this element is a cycle, a fact which reduces to the identity $\overline{s}_\alpha - \overline{s}_\gamma = (\overline{s}_\alpha - \overline{s}_\beta) + (\overline{s}_\beta - \overline{s}_\gamma)$ on $U_\alpha \cap U_\beta \cap U_\gamma$. By hypothesis, $\Gamma_U \mathcal{C}^\bullet(A, \mathcal{F})$ is acyclic, so our cycle is a boundary. That is, there exist sections $t_\alpha \in A(U_\alpha)$ such that on each $U_\alpha \cap U_\beta$ we have $\overline{s}_\alpha - \overline{s}_\beta = t_\alpha - t_\beta$, or equivalently $\overline{s}_\alpha - t_\alpha = \overline{s}_\beta - t_\beta$.

Since the t_α are sections of A, the sections $\overline{s}_\alpha - t_\alpha \in M(U_\alpha)$ are again lifts of $s|U_\alpha$. But now they agree on intersections $U_\alpha \cap U_\beta$, so by the sheaf axiom, they are the restrictions $\overline{s}|U_\alpha$ of a secton $\overline{s} \in M(U)$, which is our desired lift of s. □

Lemma 2.5. *If A is $\mathcal{B}$-acyclic and $U \in \mathcal{B}$, then $(j_U)_* \, j_U^{-1} A$ is $\mathcal{B}$-acyclic.*

Proof. This follows easily from the canonical identifications $\Gamma_V (j_U)_* \, j_U^{-1} A = \Gamma_V (j_{U \cap V})_* \, j^{-1}_{U \cap V} A$ and the fact that $\mathcal{B}$ is closed under finite intersections. □

To make sense of the statement of the next corollary, note that for fixed $\mathcal{F}$, the Čech complex $\mathcal{C}^\bullet(M, \mathcal{F})$ is functorial in M.

Corollary 2.6. *Let $\mathcal{F}$ be a cover in $\mathcal{B}$ of $U \in \mathcal{B}$. If $0 \to A \to M \to N \to 0$ is exact and A is $\mathcal{B}$-acyclic, then the sequence of complexes*

$$0 \to \Gamma_V \mathcal{C}^\bullet(A, \mathcal{F}) \to \Gamma_V \mathcal{C}^\bullet(M, \mathcal{F}) \to \Gamma_V \mathcal{C}^\bullet(N, \mathcal{F}) \to 0 \tag{5}$$

is exact for every $V \in \mathcal{B}$.

Proof. We are to show that $0 \to \Gamma_V \mathcal{C}^n(A, \mathcal{F}) \to \Gamma_V \mathcal{C}^n(M, \mathcal{F}) \to \Gamma_V \mathcal{C}^n(N, \mathcal{F}) \to 0$ is exact for all n. Since Γ_V commutes with products, this reduces to proving exactness of $0 \to \Gamma_V (j_{U_I})_* \, j^{-1}_{U_I} A \to \Gamma_V (j_{U_I})_* \, j^{-1}_{U_I} M \to \Gamma_V (j_{U_I})_* \, j^{-1}_{U_I} N \to 0$ for the open sets U_I in the definition of $\mathcal{C}(-, \mathcal{F})$. Since $\mathcal{B}$ is closed under finite intersections, we have $U_I \in \mathcal{B}$. The result now follows from Lemmas 2.4 and 2.5. □

Remark 2.7. Since $\mathcal{B}$ is a base of the topology, Corollary 2.6 implies that $0 \to \mathcal{C}^\bullet(A, \mathcal{F}) \to \mathcal{C}^\bullet(M, \mathcal{F}) \to \mathcal{C}^\bullet(N, \mathcal{F}) \to 0$ is an exact sequence of complexes of sheaves. This statement is weaker than (5), however, since Γ_V is not an exact functor.

Corollary 2.8. *If $0 \to A \to B \to N \to 0$ is exact and both A and B are $\mathcal{B}$-acyclic, then N is $\mathcal{B}$-acyclic.*

Proof. The hypothesis is that for every cover $\mathcal{F}$ in $\mathcal{B}$ of some $U \in \mathcal{B}$, the first two terms in (5) are acyclic complexes for $V = U$. The conclusion is that the same holds for the third term. □

Now recall from Proposition 5.6 in the Notes on Derived Categories and Derived Functors, that a sufficient condition for resolutions in a class of sheaves $\mathfrak{A}$ to compute the derived functor RF of a left exact functor F on $\mathbf{D}^+(\mathcal{O}_X\text{-Mod})$ are: (i) every sheaf M of $\mathcal{O}_X$ modules is a subsheaf of a sheaf $A \in \mathfrak{A}$; (ii) $\mathfrak{A}$ is closed under finite direct sums, and if $0 \to A \to B \to N \to 0$ is exact and $A, B \in \mathfrak{A}$, then $N \in \mathfrak{A}$; and (iii) the functor F is exact on all exact sequences $0 \to A \to M \to N \to 0$ with $A \in \mathfrak{A}$.

In Corollary 2.3, Lemma 2.4 and Corollary 2.8 we have shown that the class $\mathfrak{A}$ of $\mathcal{B}$-acyclic sheaves has these properties with respect to the global section functor, and more generally with respect to the functor of sections Γ_U for all $U \in \mathcal{B}$. This proves the following.

Proposition 2.9. *The $\mathcal{B}$-acyclic sheaves are acyclic for the global section functor Γ_X and more generally for the functor of sections Γ_U for $U \in \mathcal{B}$. In particular, the derived functors $R\Gamma_X$ and $R\Gamma_U$ for $U \in \mathcal{B}$ can be calculated using $\mathcal{B}$-acyclic resolutions.*

3. Finiteness

In §2 the space X and base $\mathcal{B}$ were arbitrary. We now explore to what degree the theory can be improved under the following extra assumption.

(C) Every member of $\mathcal{B}$, in particular X itself, is quasi-compact.

Assumption (C) implies that every cover $\mathcal{F}$ in $\mathcal{B}$ of an open set $U \in \mathcal{B}$ has a finite subcover $\mathcal{F}' \subseteq \mathcal{F}$. Conversely, since $\mathcal{B}$ is a base of the topology, this condition implies (C).

In the presence of (C) we study a class of sheaves defined by a weaker condition than 2.1.

Definition 3.1. A sheaf of $\mathcal{O}_X$ modules M on X is *finitely $\mathcal{B}$-acyclic* if the complex $\Gamma_U \mathcal{C}^\bullet(M, \mathcal{F})$ is acyclic for every $U \in \mathcal{B}$ and every *finite* covering $\mathcal{F}$ of U by subsets $U \supseteq U_\alpha \in \mathcal{B}$.

Of course, of course $\mathcal{B}$-acyclic sheaves are finitely $\mathcal{B}$-acyclic, so Lemma 2.2 immediately implies:

Corollary 3.2. *Flasque sheaves are finitely $\mathcal{B}$-acyclic.*

Corollary 3.3. *Every sheaf of $\mathcal{O}_X$ modules is a subsheaf of a finitely $\mathcal{B}$-acyclic sheaf.*

Assuming (C), we can choose the local lifts $\overline{s}_\alpha$ in the proof of Lemma 2.4 on a finite covering $\mathcal{F}$ of U in $\mathcal{B}$. Then the proof goes through assuming only that A is finitely B-acyclic, to give the following result.

Lemma 3.4. *Assume (C) holds. If $0 \to A \to M \to N \to 0$ is an exact sequence of sheaves and A is finitely $\mathcal{B}$-acyclic, then $0 \to A(U) \to M(U) \to N(U) \to 0$ is exact for all $U \in \mathcal{B}$. In particular, this holds for $U = X$.*

The same reasoning as in the proof of Lemma 2.5 also proves the finite version, and the proof of Corollary 2.6 applies verbatim with Lemmas 3.4 and 3.5 in place of 2.4 and 2.5.

Lemma 3.5. *If A is finitely $\mathcal{B}$-acyclic and $U \in \mathcal{B}$, then $(j_U)_* j_U^{-1} A$ is finitely $\mathcal{B}$-acyclic.*

Corollary 3.6. *Assume (C) holds. Let $\mathcal{F}$ be a finite cover in $\mathcal{B}$ of $U \in \mathcal{B}$. If $0 \to A \to M \to N \to 0$ is exact and A is finitely $\mathcal{B}$-acyclic, then the sequence of complexes*

$$0 \to \Gamma_V \mathcal{C}^\bullet(A, \mathcal{F}) \to \Gamma_V \mathcal{C}^\bullet(M, \mathcal{F}) \to \Gamma_V \mathcal{C}^\bullet(N, \mathcal{F}) \to 0 \tag{6}$$

is exact for every $V \in \mathcal{B}$.

Corollary 3.7. *Assume (C) holds. If $0 \to A \to B \to N \to 0$ is exact and both A and B are $\mathcal{B}$-acyclic, then N is $\mathcal{B}$-acyclic.*

Proof. Same as the proof of Corollary 2.8, but considering only finite covers. □

Corollary 3.3, Lemma 3.4 and Corollary 3.7 now allow us to deduce the finite version of Proposition 2.9.

Proposition 3.8. *Assume (C) holds. The finitely $\mathcal{B}$-acyclic sheaves are acyclic for the global section functor Γ_X and more generally for the functor of sections Γ_U for $U \in \mathcal{B}$. In particular, the derived functors $R\Gamma_X$ and $R\Gamma_U$ for $U \in \mathcal{B}$ can be calculated using finitely $\mathcal{B}$-acyclic resolutions.*

4. Sheaf cohomology on affine schemes

In this section we will prove the following fundamental theorem.

Theorem 4.1. *Every quasi-coherent sheaf on an affine scheme $X = \operatorname{Spec}(R)$ is acyclic for the global section functor Γ.*

Before turning to the proof, let's try to understand where this theorem fits in with our other knowledge about affine schemes. Recall that if $X = \operatorname{Spec}(R)$, we have an equivalence of categories between R modules and quasi-coherent sheaves on X, given in one direction by the construction of the sheaf $\widetilde{M}$ associated to an R module M and in the other by Γ. Hence Γ is an exact functor on the category of quasi-coherent sheaves. At first sight Theorem 4.1 might seem to be merely a restatement of this fact. What the latter actually means, however, is that $R\Gamma = \Gamma$ as a functor on the derived category $\mathbf{D}(\operatorname{Qco}(X))$ of the category of quasi-coherent sheaves on X. Theorem 4.1 is the stronger statement that the derived functor $R\Gamma$ on the derived category of *all* sheaves of $\mathcal{O}_X$ modules, or indeed on the derived category of all sheaves of abelian groups, has the property that $R\Gamma Q = \Gamma Q$ for Q a quasi-coherent $\mathcal{O}_X$ module. In this context, the definition of $R\Gamma$ is a purely topological one, in which the quasi-coherent sheaves play no special role *a priori*.

Nevertheless, since we mainly care about quasi-coherent sheaves, why not simply work with $\mathbf{D}(\operatorname{Qco}(X))$ and avoid the need for Theorem 4.1? The answer is that in general the derived category $\mathbf{D}(\mathcal{O}_X\text{-Mod})$ provides a better context for sheaf cohomology than $\mathbf{D}(\operatorname{Qco}(X))$, which has various shortcomings, among them the lack of enough injective or flasque objects in $\operatorname{Qco}(X)$, and the fact that direct image functors f_* need not preserve quasi-coherent sheaves.

Proof of Theorem 4.1. Consider the base $\mathcal{B}$ for the topology on X consisting of open sets $X_f = X \setminus V(f)$. It is closed under intersections since $X_f \cap X_g = X_{fg}$. Since X_f is homeomorphic to $\operatorname{Spec}(R_f)$, it is quasi-compact, so $\mathcal{B}$ has property (C) from §3.

Every quasi-coherent sheaf $\mathcal{M}$ on X is isomorphic to $\widetilde{M}$, where $M = \mathcal{M}(X)$. By Proposition 3.8, it suffices to prove that $\widetilde{M}$ is finitely $\mathcal{B}$-acyclic for every R module M. Since the members of $\mathcal{B}$ are themselves affine schemes, we need only consider finite covers in $\mathcal{B}$ of $U = X$. That is, we are given finitely many elements $f_i \in R$ which generate the unit ideal, so the $U_i = X_{f_i}$ cover $\operatorname{Spec}(R)$. For this cover $\mathcal{F}$, we are to prove that $\Gamma_X \mathcal{C}^\bullet(\widetilde{M}, \mathcal{F})$ is acyclic.

Now, the products in the definition (2) of $\mathcal{C}^n(\widetilde{M}, \mathcal{F})$ are finite, so they are direct sums. Consider a summand $(j_{U_I})_* j_{U_I}^{-1} \widetilde{M}$, in which $U_I = X_g$, with $g = \prod_{i \in I} f_i$. Then $j_{U_I}^{-1}\widetilde{M} = \widetilde{M_g}$ as a sheaf on $X_g = \operatorname{Spec}(R_g)$, and its direct image under the morphism of affine schemes $j_{U_I} \colon X_g \hookrightarrow X$ is $(j_{U_I})_* j_{U_I}^{-1}\widetilde{M} = \widetilde{M_g}$, where we regard M_g as an R module. This shows that $\mathcal{C}^\bullet(\widetilde{M}, \mathcal{F})$ is a complex of quasi-coherent sheaves. It is acyclic by Corollary 1.4. Since Γ_X is exact on quasi-coherent sheaves, $\Gamma_X \mathcal{C}^\bullet(\widetilde{M}, \mathcal{F})$ is acyclic. □

Corollary 4.2. *Let $f \colon X \to Y$ be an affine morphism. If M is a quasi-coherent sheaf of $\mathcal{O}_X$ modules, then $Rf_*M = f_*M$.*

Proof. Theorem 4.1 implies that M satisfies the hypothesis of Proposition 5.12 in the Notes on Derived Categories and Derived Functors, with $\mathcal{B}$ the set of all open affine subsets of Y. □

Lemma 4.3. *The derived functor $R\Gamma_X$ for sheaves of $\mathcal{O}_X$ modules on any ringed space X commutes with arbitrary products.*

Proof. Products of injective objects in any abelian category $\mathcal{A}$ are injective. We can therefore compute $R\Gamma_X(\prod_\alpha M_\alpha)$ by taking an injective resolution $M \underset{\text{qis}}{\to} I^\bullet_\alpha$ of each M_α and applying Γ_X to $\prod_\alpha I^\bullet_\alpha$. Since Γ_X commutes with products, the result coincides with $\prod_\alpha R\Gamma(M_\alpha)$. □

Remark 4.4. The proof shows that if F is any left exact functor which commutes with products, then so does RF.

Corollary 4.5. *Any product of quasi-coherent sheaves on an affine scheme is acyclic for the global sections functor.*

For any collection $(f_\alpha)_{\alpha \in \mathcal{I}}$ of elements $f_\alpha \in R$, the open sets $\mathcal{F} = (X_{f_\alpha})$ cover $U = X \setminus V((f_\alpha))$. If M is an R module, then $C^\bullet(M, (f_\alpha)) \underset{\text{def}}{=} \Gamma_X \mathcal{C}^\bullet(\widetilde{M}, \mathcal{F})$ is the 'local cohomology complex' with n-th term

$$C^n(M, (f_\alpha)) = \prod_{I \subseteq \mathcal{I},\, |I| = n} M_{f_I}, \tag{7}$$

where $f_I = \prod_{\alpha \in I} f_\alpha$. The differentials are constructed from projections on the factors and the canonical homomorphisms $M_{f_J} \to M_{f_I}$ for $J \subseteq I$, with signs depending on a chosen total ordering of $\mathcal{I}$ in the same way as in Definition 1.2.

Given an index $\alpha \in \mathcal{I}$, let $\mathcal{I}' = \mathcal{I} \setminus \{\alpha\}$ and $(f_\beta)' = (f_\beta)_{\beta \in \mathcal{I}'}$. As in Lemma 1.3, the local cohomology complex is the -1 shift of the mapping cone of a homomorphism

$$C^\bullet(M, (f_\beta)') \to C^\bullet(M_{f_\alpha}, (f_\beta)'). \tag{8}$$

However, if $\mathcal{I}$ is infinite, (8) does not reduce to an isomorphism upon localizing at $\mathfrak{p} \in X_{f_\alpha}$, because localization does not commute with infinite products.

It is nevertheless true that if the f_α generate the unit ideal, the complex $C^\bullet(M, (f_\alpha))$ is acyclic, even in the infinite case. In other words, the quasi-coherent sheaf $\widetilde{M}$ is in fact $\mathcal{B}$-acyclic and not just finitely $\mathcal{B}$-acyclic. This follows from Corollary 4.5, since $\mathcal{C}^\bullet(\widetilde{M}, \mathcal{F})$ is a complex of products of quasi-coherent sheaves.

5. Čech cohomology

Let X be a scheme and $\mathcal{F} = (U_\alpha)_{\alpha \in \mathcal{I}}$ a covering of X by open affine subschemes U_α. Suppose that *for every finite, non-empty $I \subseteq \mathcal{I}$, the intersection $U_I = \bigcap_{\alpha \in I} U_\alpha$ is affine.* In particular, this condition holds automatically if X is separated, since the intersection of any two affines in X is then affine.

Let M be a quasi-coherent sheaf of $\mathcal{O}_X$ modules. As in Remark 1.5, define $\widehat{\mathcal{C}}^\bullet(M, \mathcal{F})$ to be the truncation of $\mathcal{C}^\bullet(M, \mathcal{F})[1]$ to terms in non-negative degree, that is, we drop the term M in degree -1.

Theorem 5.1. *Under the assumptions above,*

$$R\Gamma(M) \cong \Gamma\widehat{\mathcal{C}}^\bullet(M, \mathcal{F}). \tag{9}$$

In particular, $H^i(X, M) \underset{def}{=} R^i\Gamma(M)$ is the i-th cohomology of the complex $\Gamma\widehat{\mathcal{C}}^\bullet(M, \mathcal{F})$.

Remark 5.2. We can describe the complex $\Gamma\widehat{\mathcal{C}}^\bullet(M, \mathcal{F})$ explicitly. For each non-empty finite subset $I \subseteq \mathcal{I}$, U_I is an affine scheme $\operatorname{Spec}(R_I)$, $M(U_I)$ is an R_I module, and $\Gamma\widehat{\mathcal{C}}^n(M, \mathcal{F})$ is the product of the $M(U_I)$ for all I of cardinality $n+1$. Of course it is not always easy to calculate the cohomology of a complex even if one has a fully explicit description of its terms and differentials.

Proof. By Corollary 1.4, the complex $\widehat{\mathcal{C}}^\bullet(M, \mathcal{F})$ is quasi-isomorphic to M, so it is enough to show that each $\widehat{\mathcal{C}}^n(M, \mathcal{F}) = \mathcal{C}^{n+1}(M, \mathcal{F})$ is acyclic for Γ. By Lemma 4.3, this reduces to each $(j_{U_I})_* j_{U_I}^{-1} M$ being acyclic for Γ. Since $U_I \cap U_\alpha$ is affine for all $\alpha \in \mathcal{I}$, j_{U_I} is an affine morphism. Hence $(j_{U_I})_* j_{U_I}^{-1} M = R(j_{U_I})_* j_{U_I}^{-1} M$, by Corollary 4.2. Then $R\Gamma(j_{U_I})_* j_{U_I}^{-1} M = R\Gamma R(j_{U_I})_* j_{U_I}^{-1} M = R\Gamma(U_I, M|U_I)$ by Corollary 5.11 in the Notes on Derived Categories and Derived Functors. Finally, $R\Gamma(U_I, M|U_I) = \Gamma(U_I, M|U_I)$ by Theorem 4.1, since U_I is affine. This shows that $R^i\Gamma(j_{U_I})_* j_{U_I}^{-1} M = 0$ for $i > 0$, that is, $(j_{U_I})_* j_{U_I}^{-1} M$ is acyclic for Γ. ☐

Corollary 5.3. *Under the same assumptions as in Theorem 5.1, except with $U = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$ not necessarily equal to X, we have*

$$R\Gamma(U, M) \cong \Gamma\widehat{\mathcal{C}}^\bullet(M, \mathcal{F}). \tag{10}$$

Proof. Since $U_I \subseteq U$ for all non-empty $I \subseteq \mathcal{I}$, we have $\Gamma_X (j_{U_I})_* j_{U_I}^{-1} M = \Gamma_U (j_{U_I})_* j_{U_I}^{-1} M$. Since Γ_X and Γ_U commute with products, it follows that $\Gamma_X \widehat{\mathcal{C}}^\bullet(M, \mathcal{F}) = \Gamma_U \widehat{\mathcal{C}}^\bullet(M, \mathcal{F})$. Then the result follows from Theorem 5.1 applied to U instead of X. $\square$

As a particular case of Corollary 5.3, suppose $X = \operatorname{Spec}(R)$ is affine and $U = X \setminus V(\mathfrak{a})$ is any open subset. If $(f_\alpha)_{\alpha \in \mathcal{I}}$ generate the ideal $\mathfrak{a}$, then the open sets $U_\alpha = X_{f_\alpha}$ form a covering of U satisying the hypotheses. Given any R module M, we can therefore calculate $R\Gamma(U, \widetilde{M})$ as a truncation of the shifted local cohomology complex $C^\bullet(M, (f_\alpha))[1]$. In particular, up to isomorphism in the derived category of R modules, this complex depends only on M and $\mathfrak{a}$, and not on the choice of generators f_α.