10. Jacobson preschemes

10.1. Very dense subsets of a topological space.

(10.1.1) A subset $T$ of a topological space $X$ is quasi-constructible if $T$ is a finite union of locally closed subsets. $T$ is locally quasi-constructible if every $x \in X$ has an open neighborhood $V$ such that $T \cap V$ is quasi-constructible in $V$. The two notions are equivalent if $X$ is quasi-compact. Let $\mathcal{Qc}(X)$, $\mathcal{Lqc}(X)$ denote the set of (locally) quasi-constructible subsets. Then $\mathcal{Qc}(X)$ and $\mathcal{Lqc}(X)$ are closed under finite intersections, unions, and complements, and preimages via continuous maps. Let $\mathcal{O}(X)$ denote the set of open subsets of $X$, $\mathcal{Cl}(X)$ the set of closed subsets.

Proposition (10.1.2). — Let $X_0$ be a subspace of $X$. The following are equivalent.

(a) For every non-empty locally closed $Z \subseteq X$, $Z \cap X_0 \neq \emptyset$.
(b) For every non-empty locally quasi-constructible $Z \subseteq X$, $Z \cap X_0 \neq \emptyset$.
(c) $U \mapsto f^{-1}(U)$ from $\mathcal{O}(X)$ to $\mathcal{O}(X_0)$ is injective (hence bijective).
(d) $Z \mapsto Z \cap X_0$ from $\mathcal{Cl}(X)$ to $\mathcal{Cl}(X_0)$ is injective (hence bijective).

Definition (10.1.3). — When the conditions in (10.1.2) hold, we say that $X_0$ is very dense in $X$.

Corollary (10.1.4). — If $X_0$ is very dense in $X$, and $U \subseteq X$ is open, then $U \cap X_0$ is very dense in $U$. Conversely, if $X = \bigcup_\alpha U_\alpha$ is an open covering such that $U_\alpha \cap X_0$ is very dense in $U_\alpha$ for each $\alpha$, then $X_0$ is very dense in $X$.

10.2. Quasi-homeomorphisms.

Proposition (10.2.1). — Let $f: X_0 \to X$ be a continuous map. The following are equivalent.

(a) $U \mapsto f^{-1}(U)$ from $\mathcal{O}(X)$ to $\mathcal{O}(X_0)$ is bijective.
(b) The topology on $X_0$ is the inverse image of that on $X$, and $f(X_0)$ is very dense in $X$.
(c) The functor $f^{-1}$ from sheaves on $X$ to sheaves on $X_0$ is an equivalence of categories (this holds both for sheaves of sets and for sheaves of abelian groups).

Definition (10.2.2). — A map $f$ satisfying the conditions in (10.2.1) is a quasi-homeomorphism. In particular, by (10.2.1, b), a subspace $X_0 \subseteq X$ is very dense if and only if the inclusion $X_0 \hookrightarrow X$ is a quasi-homeomorphism.

Corollary (10.2.3). — The composite of two quasi-homeomorphisms is a quasi-homeomorphism.
Corollary (10.2.4). — If \( f : X \to Y \) is a quasi-homeomorphism, \( Y' \subseteq Y \) is locally quasi-constructible, and \( X' = f^{-1}(Y') \), then the restriction \( f' = (f|X') : X' \to Y' \) is a quasi-homeomorphism.

Corollary (10.2.5). — Let \( f : X \to Y \) be a continuous map, \( Y = \bigcup \alpha V_\alpha \) an open covering. If the restriction \( f^{-1}(V_\alpha) \to V_\alpha \) of \( f \) is a quasi-homeomorphism for all \( \alpha \), then \( f \) is a quasi-homeomorphism.

Corollary (10.2.6). — Let \( f : X \to Y \) be a quasi-homeomorphism, \( Y' \subseteq Y \) locally quasi-constructible, \( X' = f^{-1}(Y') \). Then \( Y' \) is quasi-compact (resp. Noetherian, retro-compact) iff \( X' \) is.

Proposition (10.2.7). — Let \( f : X \to Y \) be a quasi-homeomorphism. Then the map \( Z \mapsto f^{-1}(Z) \) from subsets of \( Y \) to subsets of \( X \) induces bijections between the open, closed, locally closed, quasi-constructible, locally quasi-constructible, constructible, and locally constructible subsets of \( X \) and \( Y \).

Remarks (10.2.8). — (i) If \( f : X \to Y \) is a quasi-homeomorphism then for any sheaf of abelian groups \( \mathcal{F} \) on \( Y \), the canonical functorial map

\[
(10.2.8.1) \quad \Gamma(Y, \mathcal{F}) \to \Gamma(X, f^{-1}(\mathcal{F}))
\]

is an isomorphism. Since \( f^{-1} \) is exact, it follows that the canonical maps in cohomology

\[
H^i(Y, \mathcal{F}) \to H^i(X, f^{-1}(\mathcal{F}))
\]

are isomorphisms.

(ii) If \( f \) is a quasi-homeomorphism then \( f^{-1} \) gives an equivalence of categories of sheaves of rings on \( X \) and on \( Y \). If \( f = (\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) is a morphism of ringed spaces, \( \psi \) is a quasi-homeomorphism, and \( \theta^\sharp \) is an isomorphism, then \( f^* = f^{-1} \) gives an equivalence of categories between sheaves of \( \mathcal{O}_Y \) modules and sheaves of \( \mathcal{O}_X \) modules. This extends to isomorphisms of Ext functors, and more generally to equivalences between all the usual functorial constructions involving sheaves and cohomology on the two spaces \( X \) and \( Y \).

The 1971 revised edition of EGA I by Grothendieck and Dieudonné, which also includes material from EGA IV, §10, adds to the above the following definition and results.

Definition. A topological space \( X \) is sober if every irreducible closed subset \( Z \subseteq X \) has a unique generic point, that is, a point \( z \) such that \( Z = \{z\} \) (0, 2.1.2).

Every prescheme \( X \) is sober (I, 2.1.5).

For any space \( X \), let \( X^+ \) denote the set of irreducible closed subsets of \( X \). If \( V \subseteq X \) is closed, then \( V^+ \) is a subset of \( X^+ \). The correspondence \( V \mapsto V^+ \) preserves finite unions and arbitrary intersections, making the subsets \( V^+ \) the closed subsets of a topology on \( X^+ \).

The map \( j : X \to X^+ \) defined by \( j(z) = \{z\} \) is continuous, and \( j^{-1}(V^+) = V \). It follows that \( V \to V^+ \) is a bijection from closed subsets of \( X \) to closed subsets of \( X^+ \) and \( j^{-1} \) induces its inverse. Hence \( j \) is a quasi-homeomorphism.
The space $X^+$ is sober. Its irreducible closed subsets are exactly the sets $Z^+$ for $Z \in X^+$, and we have $Z^+ = \{Z\}$.

Given a continuous map $f : X \to Y$, there is a unique continuous map $f^+ : X^+ \to Y^+$ such that $f^+j_X = j_Yf$. This makes $(-)^+$ a functor from topological spaces to sober spaces. Every continuous map $f : X \to Y$, where $Y$ is sober, factors uniquely through $j : X \to X^+$. This implies that $(-)^+$ is left adjoint to the inclusion of sober spaces into topological spaces.

Every quasi-homeomorphism between sober spaces is a homeomorphism. It follows that a continuous map $f : X \to Y$ is a quasi-homeomorphism if and only if $f^+$ is a homeomorphism. One can therefore view sober spaces as canonical representatives of topological spaces up to quasi-homeomorphism.


Definition (10.3.1). — A topological space $X$ is Jacobson if the set of closed points $X_0$ of $X$ is very dense in $X$; that is, if $X_0 \hookrightarrow X$ is a quasi-homeomorphism.

Proposition (10.3.2). — Let $X$ be Jacobson, $Z \subseteq X$ locally quasi-constructible. Then the subspace $Z$ is Jacobson, and a point $z \in Z$ is closed in $Z$ iff it is closed in $X$.

Proposition (10.3.3). — Let $X = \bigcup \alpha U_\alpha$ be an open covering. Then $X$ is Jacobson iff every $U_\alpha$ is Jacobson.

10.4. Jacobson preschemes and Jacobson rings.

Definition (10.4.1). — A prescheme $X$ is Jacobson if its underlying topological space is Jacobson. A ring $A$ is Jacobson if $\text{Spec}(A)$ is Jacobson.

According to this definition, $A$ is Jacobson if and only if every radical ideal of $A$ is an intersection of maximal ideals; if and only if every prime ideal of $A$ is an intersection of maximal ideals (the latter is the usual definition of a Jacobson ring).

Proposition (10.4.2). — Let $X = \bigcup \alpha U_\alpha$ be an open affine covering of the prescheme $X$. Then $X$ is Jacobson iff each ring $\mathcal{O}_X(U_\alpha)$ is Jacobson.

(10.4.3). Examples: a discrete space is Jacobson, hence an Artinian ring is Jacobson. A principal ideal domain with infinitely many maximal ideals (such as $\mathbb{Z}$) is Jacobson. A Noetherian local ring is Jacobson iff its maximal ideal is its only prime ideal; that is, iff it is Artinian. By (10.3.2), any sub-prescheme of a Jacobson scheme is Jacobson.

Proposition (10.4.4). — Let $B$ be an integral domain. The following are equivalent.

(a) There exists $f \neq 0$ in $B$ such that $B_f$ is a field.
(b) The field of fractions of $B$ is a finitely generated $B$-algebra.
(c) There exists a field $K$ containing $B$, which is a finitely generated $B$-algebra.
(d) The generic point of $\text{Spec}(B)$ is isolated (i.e., the set consisting of only that point is open).

(d) $\iff$ (a) $\iff$ (b) $\Rightarrow$ (c) are easy. The significant point is that (c) implies the others, which is a version of Hilbert’s Nullstellensatz.
Proposition (10.4.5). — Given a ring $A$, the following are equivalent.

(a) $A$ is Jacobson.

(b) For every non-maximal prime ideal $p \subseteq A$ and every $f \neq 0$ in $B = A/p$, $B_f$ is not a field.

$b'$ Every finitely generated $A$-algebra $K$ which is a field, is finite over $A$ (i.e., finitely generated as an $A$-module; thus a finite algebraic extension of $A/m$, where $m$ is a maximal ideal).

Corollary (10.4.6). — Every algebra $B$ of finite type over a Jacobson ring $A$ is Jacobson. Moreover, the preimage in $A$ of any maximal ideal of $B$ is maximal. In particular, any finitely generated algebra over $\mathbb{Z}$ or a field is Jacobson.

Corollary (10.4.7). — If $X$ is a Jacobson prescheme and $f : Y \rightarrow X$ is a morphism locally of finite type, then $Y$ is Jacobson, and $f$ maps every closed point of $X$ to a closed point of $Y$. [Moreover, if $f(x) = y$, then $k(x)$ is a finite algebraic extension of $k(y)$.

Corollary (10.4.8). — If $X$ is locally of finite type over an algebraically closed field $k$, then the $k$-rational points of $X$ are very dense in $X$.

Indeed, the $k$-rational points are the closed points, by (I, 6.4.2), and $X$ is Jacobson.

(10.4.9–11). A number of questions in algebraic geometry can be reduced to the case of a finitely generated algebra over $\mathbb{Z}$ or a field, so the fact that such rings are Jacobson is particularly important. EGA gives two applications, of which the second is the following.

Proposition: Let $X$ be an $S$-prescheme of finite type. Then any universally injective $S$-morphism $g : X \rightarrow X$ is bijective.

[The morphism $g$ is universally injective if it induces an injection $X(K) \rightarrow X(K)$ for every field $K$.]

In fact, it is shown in (IV, 17.9.7) that under the hypotheses of the Proposition, $g$ is an isomorphism.