Synopsis of material from EGA Chapter IV, §1.1–1.7

1. Relative finiteness condtions. Constructible subsets of preschemes.

Some of the concepts to follow were introduced in Chapter I, §6, but are given a more compete treatment here.

1.1. **Quasi-compact morphisms.**

Definition (1.1.1). — A morphism $f\colon X \to Y$ is *quasi-compact* if $f^{-1}(U)$ is quasi-compact for every quasi-compact open subset $U \in Y$.

If B is a base of the topology on Y consisting of open affines, then f is quasi-compact if and only if $f^{-1}(V)$ has a finite covering by open affines for all $V \in B$.

If f is quasi-compact, then so is its restriction to $f^{-1}(V)$, for every open $V \subseteq Y$. Conversely, if Y has a covering by open sets U_α such that each restriction $f^{-1}(U_\alpha) \to U_\alpha$ is quasi-compact, then f is quasi-compact.

Proposition (1.1.2). — *(i) Every closed immersion $X \to Y$ is quasi-compact. If the underlying space of Y is locally Noetherian, or that of X is Noetherian, then every immersion is quasi-compact.* [Actually, if the underlying space of X is Noetheian, then every open subset of X is quasi-compact, hence every morphism $X \to Y$ is quasi-compact.]

(ii) A composite of quasi-compact morphisms is quasi-compact.

(iii) Every base extension of a quasi-compact morphism is quasi-compact.

(iv) The produc $f \times_S g$ of quasi-compact S-morphisms is quasi-compact.

(v) Given $f\colon X \to Y$ and $g\colon Y \to Z$, if $g \circ f$ is quasi-compact, and either g is separated or the underlying space of X is locally Noetherian, then f is quasi-compact.

(vi) f is quasi-compact if and only if f_{red} is.

Proposition (1.1.3). — *Given $f\colon X \to Y$ and $g\colon Y \to Z$, if $g \circ f$ is quasi-compact, and f is surjective, then g is quasi-compact.*

This generalizes (1.1.2, vi).

Corollary (1.1.4). — *Given $f\colon X \to Y$ and $g\colon Y' \to Y$, let $X' = X \times_Y Y'$ and $f' = f_{(Y')}\colon X' \to Y'$. If g is quasi-compact and surjective, then f' quasi-compact implies f quasi-compact. If g is surjective, then f' dominant implies f dominant.*

A generic point of an irreducible component of X (minimal prime of R in the case $X = \mathrm{Spec}(R)$) is called a *maximal point* of X.

Proposition (1.1.5). — *Suppose $f\colon X \to Y$ is quasi-compact. The following are equivalent:*

a) f is dominant;

b) for every maximal point $y \in Y$, $f^{-1}(y) \neq \emptyset$;

c) for every maximal point $y \in Y$, $f^{-1}(y)$ contains a maximal point of X.

Proposition (1.1.6). — *Given $f'\colon X' \to Y$ and $f''\colon X'' \to Y$, let $X = X' \sqcup X''$ and let $f\colon X \to Y$ be the morphism which is f' on X' and f'' on X''. Then f is quasi-compact if and only if both f' and f'' are.*

1.2. **Quasi-separated morphisms.**

Definition (1.2.1). — A morphism $f\colon X \to Y$ is *quasi-separated*, or *the prescheme X is quasi-separated over Y*, if the diagonal morphisms $\Delta_f\colon X \to X \times_Y X$ is quasi-compact. A prescheme is called *quasi-separated* if it is quasi-separated over $\operatorname{Spec}(\mathbb{Z})$.

By (1.1.2, (i)), separated morphisms are quasi-separated and schemes [in EGA this means separated preschemes] are quasi-separated.

Proposition (1.2.2). — *(i) Every monomorphism of preschemes, and in particular every immersion, is quasi-separated.*
(ii) A composite of quasi-separated morphisms is quasi-separated.
(iii) Every base extension of a quasi-separated morphism is quasi-separated.
(iv) The product $f \times_S g$ of quasi-separated S-morphisms is quasi-separated.
(v) If $g \circ f$ is quasi-separated, then so is f.
(vi) If f is quasi-separated, then so is f_{red}.

Corollary (1.2.3). — *(i) If X is quasi-separated then every morphism $f\colon X \to Y$ is quasi-separated.*
(ii) If Y is quasi separated, then $f\colon X \to Y$ is quasi-separated if and only if X is quasi-separated.

Proposition (1.2.4). — *Given $f\colon X \to Y$ and $g\colon Y \to Z$, if g is quasi-separated and $g \circ f$ is quasi-compact, then f is quasi-compact.*

Proposition (1.2.5). — *Given $f\colon X \to Y$ and $g\colon Y' \to Y$, let $X' = X \times_Y Y'$ and $f' = f_{(Y')}\colon X' \to Y'$. If g is quasi-compact and surjective, and f' is quasi-separated, then so is f.*

Proposition (1.2.6). — *Let Y be covered by quasi-separated open sub-preschemes U_α. Then $f\colon X \to Y$ is quasi-separated if and only if each open prescheme $f^{-1}(U_\alpha) \to U_\alpha$ is quasi-separated.*

Proposition (1.2.7). — *Let X be covered by quasi-compact open subsets U_α. The following are equivalent:*
a) X is quasi-separated;
b) for every quasi-compact open subset $U \subseteq X$, the inclusion $U \to X$ is quasi-separated;
b') every intersection of two quasi-compact open subsets of X is quasi-compact;
c) each $U_\alpha \cap U_\beta$ is quasi-compact.

Corollary (1.2.8). — *If the underlying space of X is locally Noetherian then X is quasi-separated, and every morphism $X \to Y$ is quasi-separated.*

Proposition (1.2.9). — *Given $f'\colon X' \to Y$ and $f''\colon X'' \to Y$, let $X = X' \sqcup X''$ and let $f\colon X \to Y$ be the morphism which is f' on X' and f'' on X''. Then f is quasi-separated if and only if both f' and f'' are.*

1.3. **Morphisms locally of finite type.**

(1.3.1). Let B be a finitely generated A algebra. Recall that for any A algebra A', $B \otimes_A A'$ is a finitely generated A' algebra, and if C is a finitely generated B algebra then C is a finitely generated A algebra.

For an affine scheme $X = \operatorname{Spec}(A)$, we write $A(X) = \Gamma(X, \mathcal{O}_X) \cong A$.

Definition (1.3.2). — A morphism $f\colon X \to Y$ is *of finite type at* $x \in X$ if there exist open affine neighborhoods $x \in U \in X$ and $f(x) \in V \subseteq Y$ such that $A(U)$ is a finitely generated $A(V)$ algebra. We say f is *locally of finite type* if f is of finite type at every $x \in X$.

Proposition (1.3.3). — *If Y is locally Noetherian and $f\colon X \to Y$ is locally of finite type, then X is locally Noetherian.*

Propsition (1.3.4). — *(i) Every immersion is locally of finite type.*

(ii) A composite of morphisms locally of finite type is locally of finite type.

(iii) Every base extension of a morphism locally of finite type is locally of finite type.

(iv) The product $f \times_S g$ of S-morphisms locally of finite type is locally of finite type.

(v) If $g \circ f$ is locally of finite type, then so is f.

(vi) If f is locally of finite type, then so is f_{red}.

Corollary (1.3.5). — *If $f\colon X \to Y$ is locally of finite type, and $Y' \to Y$ is a morphism with Y' locally Noetherian, then $X \times_Y Y'$ is locally Noetherian.*

Proposition (1.3.6). — *The morphism* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *corresponding to a ring homomorphism $\phi\colon A \to B$ is locally of finite type if and only if B is a finitely generated A algebra.*

["If" is obvious, but "only if" has algebraic content.]

Proposition (1.3.7). — *A morphism $f\colon X \to Y$ locally of finite type is surjective if and only if it induces a surjective map of geometric points $X(K) \to Y(K)$ for every algebraically closed field K.*

(1.3.8). An A algebra B is called *essentially of finite type* if it is a localization $B = S^{-1}C$ of a finitely generated A algebra C.

Proposition (1.3.9). — *(i) If C is a B algebra essentially of finite type and B is an A algebra essentially of finite type, then C is an A algebra essentially of finite type.*

(ii) If B is an A algebra essentially of finite type and A' is any A algebra, then $B' = B \otimes_A A'$ is an A' algebra essentially of finite type.

(1.3.10). A *local* A algebra B essentially of finite type is always of the form $C_{\mathfrak{q}}$ for a finitely generated A algebra C and prime ideal $\mathfrak{q} \subseteq C$. Let $\mathfrak{p}$ be the preimage of $\mathfrak{q}$ in A and set $S = A \setminus \mathfrak{p}$. Then $C_{\mathfrak{q}}$ is also the local ring at a prime ideal of $S^{-1}C$, which is a finitely generated algebra over $A_{\mathfrak{p}} = S^{-1}A$. We get a local homomorphism of local rings $A_{\mathfrak{p}} \to B$, making B an $A_{\mathfrak{p}}$ algebra essentially of finite type.

Proposition (1.3.11). — *In the preceding, one can take C to be a polynomial ring $A[t_1, \ldots, t_n]$.*

1.4. **Morphisms locally of finite presentation.**

(1.4.1). A (commutative) A algebra B is *finitely presented* if it is a quotient of a polynomial ring $B = A[t_1, \dots, t_n]/I$, where I is a finitely generated ideal. If this holds, then $B \otimes_A A'$ is a finitely presented A' algebra, for any A algebra A'. If B is a finitely presented A algebra and C is a finitely presented B algebra, then C is a finitely presented B algebra. If A is Noetherian, then every finitely generated A algebra is finitely presented.

Definition (1.4.2). — A morphism $f \colon X \to Y$ is *of finite presentation at* $x \in X$ if there exist open affine neighborhoods $x \in U \in X$ and $f(x) \in V \subseteq Y$ such that $A(U)$ is a finitely presented $A(V)$ algebra. We say f is *locally of finite presentation* if f is of finite presentation at every $x \in X$.

If Y is locally Noetherian, then f is locally of finite presentation if and only if it is locally of finite type.

Proposition (1.4.3). — *(i) Every local isomorphism is locally of finite presentation.*

(ii) A composite of morphisms locally of finite presentation is locally of finite presentation.

(iii) Every base extension of a morphism locally of finite presentation is locally of finite presentation.

(iv) The product $f \times_S g$ of S-morphisms locally of finite presentation is locally of finite presentation.

(v) If $g \circ f$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.

Proposition (1.4.4). — *Let B be an A algebra of the form $A[t_1, \dots, t_n]/I$. Then B is a finitely presented A algebra if and only if I is finitely generated.*

[“If” is obvious, but “only if” has algebraic content.]

Corollary (1.4.5). — *Let $j \colon X \to Y$ be an immersion, $U \subseteq Y$ an open subset such that $j(X)$ is closed in U, and $\mathcal{J} \subseteq \mathcal{O}_U$ the quasi-coherent ideal sheaf which defines $j(X)$ as a closed subscheme of U. Then j is locally of finite presentation if and only if $\mathcal{J}$ is locally a finitely generated $\mathcal{O}_U$ module.*

Proposition (1.4.6). — *The morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *corresponding to a ring homomorphism $\phi \colon A \to B$ is locally of finite presentation if and only if B is a finitely presented A algebra.*

Proposition (1.4.7). — *let B be an A algebra finitely generated as an A module. Then B is a finitely presented A algebra if and only if it is a finitely presented A module.*

1.5. **Morphisms of finite type.**

Proposition (1.5.1). — *Given a morphism $f \colon X \to Y$ and a covering of Y by open affines U_α, the following are equivalent:*

a) f is locally of finite type and quasi-compact;

b) each $f^{-1}(U_\alpha)$ is a finite union of affines V_β such that $A(V_\beta)$ is a finitely generated $A(U_\alpha)$ algebra;

c) for every open affine $U \subseteq Y$, $f^{-1}(U)$ is a finite union of affines V_β such that $A(V_\beta)$ is a finitely generated $A(U)$ algebra;

Definition (1.5.2). — A morphism satisfying the conditions in (1.5.1) is *of finite type*.

Proposition (1.5.3). — *If Y is Noetherian and $f\colon X \to Y$ is of finite type, then X is Noetherian.*

Propsition (1.5.4). — *(i) Every quasi-compact immersion $j\colon X \to Y$ is of finite type. In particular, this holds if j is a closed immersion, or the underlying space of X is Noetherian, of that of Y is locally Noetherian.*
(ii) A composite of morphisms finite type is of finite type.
(iii) Every base extension of a morphism locally of finite type is locally of finite type.
(iv) The product $f \times_S g$ of S-morphisms locally of finite type is locally of finite type.
(v) Given $f\colon X \to Y$ and $g\colon Y \to Z$, if $g \circ f$ is of finite type, and either g is quasi-separated or the underlying space of X is locally Noetherian, then f is of finite type.
(vi) If f is of finite type, then so is f_{red}.
[EGA has "X Noetherian" in (v), but "locally Noetherian" suffices, by (1.1.2, v) and (1.3.4, v).]

Corollary (1.5.5). — *If $f\colon X \to Y$ is of finite type, and $Y' \to Y$ is a morphism with Y' Noetherian, then $X \times_Y Y'$ is Noetherian.*

Corollary (1.5.6). — *If X is of finite type over a locally Noetherian prescheme S then every S-morphism $f\colon X \to Y$ is of finite type.*

Proposition (1.5.7). — *The morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *corresponding to a ring homomorphism $\phi\colon A \to B$ is of finite type if and only if B is a finitely generated A algebra.*

1.6. **Morphisms of finite presentation.**

Definition (1.6.1). — A morphism $f\colon X \to Y$ is *of finite presentation*, or X is *of finite presentation over Y*, if the following conditions hold:
(i) f is locally of finite presentation;
(ii) f is quasi-compact (or equivalently, given (i), of finite type);
(iii) f is quasi-separated.
Condition (iii) follows if f is separated or X is locally Noetherian. If Y is locally Noetherian then X is of finite presentation if and only if it is of finite type.

Proposition (1.6.2). — *(i) Every quasi-compact immersion locally of finite presentation (in particular, every quasi-compact open immersion) is of finite presentation.*
(ii) A composite of morphisms of finite presentation is of finite presentation.
(iii) Every base extension of a morphism of finite presentation is of finite presentation.
(iv) The product $f \times_S g$ of S-morphisms of finite presentation is of finite presentation.
(v) If $g \circ f$ is of finite presentation and g is quasi-separated and locally of finite presentation, then f is of finite presentation.

If f is of finite presentation, then so is its restriction to $f^{-1}(V)$, for every open $V \subseteq Y$. Conversely, if Y has a covering by open sets U_α such that each restriction $f^{-1}(U_\alpha) \to U_\alpha$ is of finite presentation, then f is of finite presentation. In other words, this condition is local on Y.

Corollary (1.6.3). — *The morphism* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *corresponding to a ring homomorphism* $\phi\colon A \to B$ *is of finite presentation if and only if* B *is a finitely presented* A *algebra.*

Remark (1.6.4). — Condition (iii) in the definition (1.6.1) is not superfluous. Consider for example an affine scheme Y whose underlying space is not Noetherian and a non-quasi-compact open $U \subseteq Y$. Let X be the gluing of two copies of Y along U, with the map $f\colon X \to Y$ that restricts to the identity map on each copy of Y. Then f is quasi-compact (as one can see directly) and locally of finite presentation (being a local isomorphism), but not quasi-separated.

Proposition (1.6.5). — *Given* $f'\colon X' \to Y$ *and* $f''\colon X'' \to Y$*, let* $X = X' \sqcup X''$ *and let* $f\colon X \to Y$ *be the morphism which is* f' *on* X' *and* f'' *on* X''*. Then* f *is of finite presentation if and only if both* f' *and* f'' *are.*

1.7. **Improvements of earlier results.**

A number of results appearing earlier in EGA can be improved by weakening the hypotheses, for example by replacing "separated" with "quasi-separated" or "of finite type" with "locally of finite type."

I will not reproduce the full list here, except to note that one particularly important result (I, 9.2.1) can be restated more simply as follows. If f is quasi-compact and quasi-separated, then the direct image functor f_* preserves quasi-coherent sheaves.