Synopsis of material from EGA Chapter II, §4

4. Projective bundles. Ample sheaves

4.1. **Definition of projective bundles.**

Definition (4.1.1). — Let $\mathbf{S}(\mathcal{E})$ be the symmetric algebra of a quasi-coherent $\mathcal{O}_Y$-module. The *projective bundle over* Y *defined by* $\mathcal{E}$ is the Y-scheme $\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathbf{S}(\mathcal{E}))$. The twisting sheaf $\mathcal{O}(1)$ on $\mathbf{P}(\mathcal{E})$ is its *fundamental sheaf.*

If Y is affine, $\mathcal{E} = \widetilde{E}$, we also write $\mathbf{P}(E)$. If $\mathcal{E} = \mathcal{O}_Y^n$, we put $\mathbf{P}_Y^{n-1} = \mathbf{P}(\mathcal{E})$, also denoted $\mathbf{P}_A^{n-1}$ if $Y = \mathrm{Spec}(A)$.

(4.1.2). A surjective homomorphism $\mathcal{E} \to \mathcal{F}$ induces a closed immersion $j\colon Q = \mathbf{P}(\mathcal{F}) \hookrightarrow \mathbf{P}(\mathcal{E}) = P$, such that $j^*\mathcal{O}_P(n) = \mathcal{O}_Q(n)$ [(3.6.2–3)].

(4.1.3). Given a morphism $\psi\colon Y' \to Y$, we have $P' = \mathbf{P}(\psi^*\mathcal{E}) = \mathbf{P}(\mathcal{E}) \otimes_Y Y'$, and $\mathcal{O}_{P'}(n) = \mathcal{O}_P(n) \otimes_Y \mathcal{O}_{Y'}$ [(3.5.3–4)].

Proposition (4.1.4). — *If $\mathcal{L}$ is invertible, we have an isomorphism $i\colon P = \mathbf{P}(\mathcal{E}) \to \mathbf{P}(\mathcal{E} \otimes \mathcal{L}) = Q$, and $i^*\mathcal{O}_Q(n) = \mathcal{O}_P(n) \otimes_Y \mathcal{L}^{\otimes n}$ [(3.1.8 (iii)), (3.2.10)].*

(4.1.5). Let $p\colon P = \mathbf{P}(\mathcal{E}) \to Y$ be the structure morphism. Since $\mathcal{E} = \mathbf{S}(\mathcal{E})_1$, we have canonical homomorphisms $\alpha_1\colon \mathcal{E} \to p_*\mathcal{O}_P(1)$ (3.3.2) and [by (0, 4.4.3)]

$$\alpha_1^\sharp\colon p^*(\mathcal{E}) \to \mathcal{O}_P(1). \tag{4.1.5.1}$$

Proposition (4.1.6). — *The canonical homomorphism (4.1.5.1) is surjective [(3.2.4)].*

4.2. **Morphisms from a prescheme to a projective bundle.**

(4.2.1). Keep the notation of (4.1.5). Let $q\colon X \to Y$ be a Y-prescheme, $r\colon X \to P$ a Y-morphism. Then $\mathcal{L}_r = r^*\mathcal{O}_P(1)$ is an invertible sheaf on X, and we deduce from (4.1.5.1) a canonical surjection

$$\phi_r\colon q^*(\mathcal{E}) \to \mathcal{L}_r. \tag{4.2.1.1}$$

Suppose $Y = \mathrm{Spec}(A)$, $\mathcal{E} = \widetilde{E}$, $f \in E$, so $r^{-1}(D_+(f)) = X_{\phi_r^\flat(f)}$ by (2.6.3), $U = \mathrm{Spec}(B) \subseteq X_{\phi_r^\flat(f)}$. On U, r corresponds to a ring homomorphism $S_{(f)} \to B$, where $S = \mathbf{S}(E)$. We have $q^*(\mathcal{E})|U = (E \otimes_A B)^{\tilde{}}$ and $\mathcal{L}_r|U = \widetilde{L}_r$, where $L_r = S(1)_{(f)} \otimes_{S_{(f)}} B$. Then ϕ_r corresponds to $E \otimes_A B \to L_r$ given by $x \otimes 1 \mapsto (f/1) \otimes (x/f)$.

(4.2.2). Conversely, suppose given $q\colon X \to Y$, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, and a homomorphism $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$. Then we get an $\mathcal{O}_X$-algebra homomorphism $\psi\colon q^*(\mathbf{S}(\mathcal{E})) \to \mathbf{S}(\mathcal{L})$, inducing a Y-morphism $r_{\mathcal{L},\psi}\colon G(\psi) \to \mathbf{P}(\mathcal{E})$ as in (3.7.1). If ϕ is sujective, then so is ψ, and $r_{\mathcal{L},\psi}$ is defined on all of X.

Proposition (4.2.3). — *Given $q\colon X \to Y$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$, Y-morphisms $r\colon X \to \mathbf{P}(\mathcal{E})$ correspond bijectively to equivalence classes of surjective $\mathcal{O}_X$-module homomorphisms $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$ with $\mathcal{L}$ invertible, where $(\mathcal{L}, \phi)$, $(\mathcal{L}, \phi')$ are equivalent if there is an isomoprhism $\tau\colon \mathcal{L} \to \mathcal{L}'$ such that $\phi' = \tau \circ \phi$.*

Theorem (4.2.4). — *The set of Y-sections of $\mathbf{P}(\mathcal{E})$ corresponds bijectively with the set of quasi-coherent subsheaves $\mathcal{F} \subseteq \mathcal{E}$ such that $\mathcal{E}/\mathcal{F}$ is invertible.* [Special case of (4.2.3) with $X = Y$.]

If $Y = \operatorname{Spec}(k)$ this identifies the k-points of $\mathbf{P}_k^{n-1}$ with the set of codimension-1 subspaces $F \subseteq k^n$.

Remark (4.2.5). — Given a quasi-coherent sheaf $\mathcal{E}$ on Y, we can assign to each Y-prescheme $X \underset{q}{\rightarrow} Y$ the set of quasi-coherent subsheaves $\mathcal{F} \subseteq q^*(\mathcal{E})$ such that $q^*(\mathcal{E})/\mathcal{F}$ is invertible. If $\psi\colon X' \to X$ is a Y-morphism, then $\psi^*\mathcal{F}$ is a subsheaf of $(q\psi)^*\mathcal{E}$ with the same property, making this assignment a functor from Y-preschemes to sets. Proposition (4.2.3) says that $\mathbf{P}(\mathcal{E})$ represents this functor.

[EGA says at this point that we will see later how to define Grassmann schemes in a similar manner, but no later section covers this.]

Corollary (4.2.6). — *Suppose that every invertible $\mathcal{O}_Y$-module is trivial. Let $A = \Gamma(Y, \mathcal{O}_Y)$, and $V = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y)$, regarded as an A-module. Let V^* be the subset of surjections in V, A^* the group of units in A. Then the set of Y-sections of $\mathbf{P}(\mathcal{E})$ is identified with V^*/A^*.*

The hypothesis holds for any local scheme Y (I, 2.4.8). For any extension K of $k(y)$, the set of K-points of the fiber $p^{-1}(y)$ of $\mathbf{P}(\mathcal{E})$ is identified (4.1.3.1) with the projective space of codimension-1 subspaces in the vector space $\mathcal{E}(y) \otimes_{k(y)} K$, where $\mathcal{E}(y) = \mathcal{E} \otimes_{\mathcal{O}_Y} k(y) = \mathcal{E}/\mathfrak{m}_y\mathcal{E}$.

If $Y = \operatorname{Spec}(A)$ and all invertible $\mathcal{O}_Y$-modules are trivial [e.g., if A is a UFD], then when $\mathcal{E} = \mathcal{O}_Y^n$, we have $V = A^n$ in (4.2.6), V^* consists of systems $(f_1, \dots, f_n)$ which generate the unit ideal in A, and two such define the same Y-section of $\mathbf{P}_A^{n-1}$ if they differ by multipication by a unit of A.

Thus $\mathbf{P}(\mathcal{E})$ generalizes the classical concept of projective space.

Remark (4.2.7). — [Promising to give details in a future Chapter V, EGA briefly discusses here how the Picard group of invertible sheaves on $\mathbf{P}(\mathcal{E})$ is related to that of Y, and how it follows that locally the automorphism group of $\mathbf{P}(\mathcal{E})$ over Y looks like $\mathcal{A}ut(\mathcal{E})/\mathcal{O}_Y^*$.]

(4.2.8). Keep the notation of (4.2.1). If $u\colon X' \to X$ is a morphism, and $r\colon X \to P$ corresponds to $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$, then $r \circ u$ corresponds to $u^*(\phi)$.

(4.2.9). Suppose $v\colon \mathcal{E} \to \mathcal{F}$ is surjective, and let $j\colon \mathbf{P}(\mathcal{F}) \to \mathbf{P}(\mathcal{E})$ be the corresponding closed immersion (4.1.2). If $r\colon X \to \mathbf{P}(\mathcal{F})$ corresponds to $\phi\colon q^*(\mathcal{F}) \to \mathcal{L}$, then $j \circ r$ corresponds to $\phi \circ q^*(v)$.

(4.2.10). Given $\psi\colon Y' \to Y$ and $r\colon X \to P$, the base extension $r_{(Y')}\colon X_{(Y')} \to P' = \mathbf{P}(\mathcal{E}')$, where $\mathcal{E}' = \psi^*(\mathcal{E})$, corresponds to $\phi_{(Y')} = \phi \otimes_{\mathcal{O}_Y} 1_{\mathcal{O}_{Y'}}$.

4.3. **The Segre morphism.**

(4.3.1). Let $\mathcal{E}$, $\mathcal{F}$ be quasi-coherent $\mathcal{O}_Y$-modules. Set $P_1 = \mathbf{P}(\mathcal{E})$, $P_2 = \mathbf{P}(\mathcal{F})$, with structure morphisms $p_i\colon P_i \to Y$. Let $Q = P_1 \times_Y P_2$, with projections $q_i\colon Q \to P_i$. Let $\mathcal{L} = \mathcal{O}_{P_1}(1) \otimes_Y \mathcal{O}_{P_2}(1) = q_1^*(\mathcal{O}_{P_1}(1)) \otimes_{\mathcal{O}_Q} q_2^*(\mathcal{O}_{P_2}(1))$, an invertible $\mathcal{O}_Q$-module. Then $r = p_1 \circ q_1 = p_2 \circ q_2$ is the structure morphism $Q \to Y$, and the canonical surjections $p_i^*(\mathcal{E}) \to$

$\mathcal{O}_{P_i}(1)$ give rise to a surjection

(4.3.1.1) $$s\colon r^*(\mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{F})\to\mathcal{L}.$$

By (4.2.2) this induces a morphism, the *Segre morphism*

(4.3.1.2) $$\zeta\colon \mathbf{P}(\mathcal{E})\times_Y\mathbf{P}(\mathcal{F})\to\mathbf{P}(\mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{F}).$$

Set $P=\mathbf{P}(\mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{F})$. Making things explicit for Y affine, $\mathcal{E}=\widetilde{E}$, $\mathcal{F}=\widetilde{F}$, one shows that

$$\zeta^{-1}(P_{x\otimes y})=(P_1)_x\times_Y(P_2)_y,$$

which comes down to the following easy lemma.

Lemma (4.3.2). — *Given A-algebras B, B', and elements $t\in B$, $t\in B'$, one has $D(t\otimes t')=D(t)\times_Y D(t')$ in* $\operatorname{Spec}(B)\times_A\operatorname{Spec}(B')$.

Proposition (4.3.3). — *The Segre morphism is a closed immersion.*

(4.3.4). The Segre morphism is functorial with respect to closed immersions $\mathbf{P}(\mathcal{E}')\hookrightarrow\mathbf{P}(\mathcal{E})$, $\mathbf{P}(\mathcal{F}')\hookrightarrow\mathbf{P}(\mathcal{F})$ induced by surjections $\mathcal{E}\to\mathcal{E}'$, $\mathcal{F}\to\mathcal{F}'$.
(4.3.5). The Segre morphism commutes with base extension by $\psi\colon Y'\to Y$.

Remark (4.3.6). — There is also a canonical closed immersion of the disjoint union $\mathbf{P}(\mathcal{E})\coprod\mathbf{P}(\mathcal{F})$ into $\mathbf{P}(\mathcal{E}\oplus\mathcal{F})$.

4.4. **Immersions into projective bundles. Very ample sheaves.**

Proposition (4.4.1). — *Let Y be a quasi-compact scheme or a prescheme with Noetherian underlying space, $q\colon X\to Y$ a morphism of finite type, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*
(i) Let $\mathcal{S}$ be a graded quasi-coherent $\mathcal{O}_Y$-algebra, and $\psi\colon q^*(\mathcal{S})\to\mathbf{S}(\mathcal{L})$ a graded $\mathcal{O}_X$-algebra homomorphism. Then $r_{\mathcal{L},\psi}$ is an everywhere defined immersion iff there exist n and a quasi-coherent submodule $\mathcal{E}$ of finite type *in $\mathcal{S}_n$, such that the induced homomorphism $q^*(\mathcal{E})\to\mathcal{L}^{\otimes n}$ is surjective and the corresponding morphism $r\colon X\to\mathbf{P}(\mathcal{E})$ is an immersion.*
(ii) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Y$-module and $\phi\colon q^*(\mathcal{F})\to\mathcal{L}$ a surjection. Then $r_{\mathcal{L},\phi}$ is an immersion if and only if there is a quasi-coherent sub-sheaf $\mathcal{E}\subseteq\mathcal{F}$ of finite type *such that $\phi'\colon q^*(\mathcal{E})\to\mathcal{L}$ is surjective and $r_{\mathcal{L},\phi'}$ is an immersion.*
[The proof uses (3.8.5).]

Definition (4.4.2). — Given $q\colon X\to Y$, an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is *very ample* (for q) if there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ and an immersion of Y-schemes $i\colon X\hookrightarrow P=\mathbf{P}(\mathcal{E})$ such that $\mathcal{L}\cong i^*\mathcal{O}_P(1)$.
Equivalently, there exists a surjection $\phi\colon q^*(\mathcal{E})\to\mathcal{L}$ such that $r_{\mathcal{L},\phi}$ is an immersion. Note that the existence of a very ample sheaf entails that q must be *separated* (3.1.3).

Corollary (4.4.3). — *If $\mathcal{L}\cong i^*\mathcal{O}_P(1)$ for an immersion $i\colon X\to P=\operatorname{Proj}(\mathcal{S})$, where $\mathcal{S}$ is a graded quasi-coherent $\mathcal{O}_Y$-algebra generated by $\mathcal{S}_1$, then $\mathcal{L}$ is very ample.*

Proposition (4.4.4). — *Suppose $q\colon X \to Y$ quasi-compact, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. The following are equivalent:*

(a) $\mathcal{L}$ is very ample for q;

(b) $q_*(\mathcal{L})$ is quasi-coherent, the canonical homomorphism $\sigma\colon q^*(q_*(\mathcal{L})) \to \mathcal{L}$ is surjective, and $r_{\mathcal{L},\sigma}\colon X \to \mathbf{P}(q_*(\mathcal{L}))$ is an immersion.

Recall that since q is quasi-compact, $q_*(\mathcal{L})$ is quasi-coherent if q is separated.

Corollary (4.4.5). — *Suppose q quasi-compact. If there exists an open covering (U_α) of Y such that $\mathcal{L}|q^{-1}(U_\alpha)$ is very ample relative to U_α, for all α, then $\mathcal{L}$ is very ample.*

Proposition (4.4.6). — *Let Y be a quasi-compact scheme or a prescheme with Noetherian underlying space, $q\colon X \to Y$ a morphism of finite type, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then the conditions of (4.4.4) are also equivalent to:*

(a′) There exists an $\mathcal{O}_Y$-module $\mathcal{E}$ of finite type *and a surjection $\phi\colon q^*(\mathcal{E}) \to \mathcal{L}$ such that $r_{\mathcal{L},\phi}$ is an immersion.*

(b′) There exists a quasi-coherent sub-$\mathcal{O}_Y$-module $\mathcal{E} \subseteq q_*(\mathcal{L})$ of finite type *with the property in (a′).*

Corollary (4.4.7). — *Suppose Y is a quasi-compact scheme or a Noetherian prescheme. If $\mathcal{L}$ is very ample for q, then there exists a graded quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{S}$, such that $\mathcal{S}_1$ is of finite type and generates $\mathcal{S}$, and an* open, dominant *Y-immersion $i\colon X \to P = \mathrm{Proj}(\mathcal{S})$ such that $\mathcal{L} \cong i^*\mathcal{O}_P(1)$.*

Proposition (4.4.8). — *Let $\mathcal{L}$ be very ample for $q\colon X \to Y$, $\mathcal{L}'$ any invertible $\mathcal{O}_X$-module such that there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ and a surjection $q^*(\mathcal{E}) \to \mathcal{L}'$. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is very ample.*

Corollary (4.4.9). — *Let $q\colon X \to Y$ be a morphism.*

(i) Given an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and invertible $\mathcal{O}_Y$-module $\mathcal{M}$, $\mathcal{L}$ is very ample if and only if $\mathcal{L} \otimes_{\mathcal{O}_X} q^*(\mathcal{M})$ is.

(ii) If $\mathcal{L}$ and $\mathcal{L}'$ are very ample, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$; in particular $\mathcal{L}^{\otimes n}$ is very ample for all $n > 0$.

Proposition (4.4.10). — *(i) Every invertible $\mathcal{O}_Y$-module $\mathcal{L}$ is very ample for the identity map $1_Y\colon Y \to Y$.*

(i′) Given $f\colon X \to Y$ and an immersion $j\colon X' \to X$, if $\mathcal{L}$ is very ample for f, then $j^*\mathcal{L}$ is very ample for $f \circ j$.

(ii) Let Z be quasi-compact, $f\colon X \to Y$ a morphism of finite type, $g\colon Y \to Z$ a quasi-compact morphism, $\mathcal{L}$ very ample for f, $\mathcal{K}$ very ample for g. Then there exists $n_0 > 0$ such that $\mathcal{L} \otimes f^*(\mathcal{K}^{\otimes n})$ is very ample for $g \circ f$, for all $n \geq n_0$.

(iii) Given $f\colon X \to Y$, $g\colon Y' \to Y$, if $\mathcal{L}$ is very ample for f, then $\mathcal{L} \otimes_Y \mathcal{O}_{Y'}$ is very ample for $f_{(Y')}$.

(iv) Given two S-morphisms $f_i\colon X_i \to Y_i$ $(i = 1, 2)$, if $\mathcal{L}_i$ is very ample for f_i, then $\mathcal{L}_1 \otimes_S \mathcal{L}_2$ is very ample for $f_1 \times_S f_2$.

(v) Given $f\colon X \to Y$, $g\colon Y \to Z$, if $\mathcal{L}$ is very ample for $g \circ f$, then $\mathcal{L}$ is very ample for f.

(vi) If $\mathcal{L}$ is very ample for $f\colon X \to Y$, then $j^*\mathcal{L}$ is very ample for f_{red}, where $j\colon X_{\mathrm{red}} \hookrightarrow X$ is the canonical injection.

[The proof of (ii) uses the following lemma, proved in §4.5]

Lemma (4.4.10.1).— *Let Z be a quasi-compact scheme or a prescheme with Noetherian underlying space, $g\colon Y \to Z$ a quasi-compact morphism, $\mathcal{K}$ very ample for g, $\mathcal{E}$ a quasi-coherent $\mathcal{O}_Y$-module of finite type. Then there exists m_0 such that for all $m \geq m_0$, $\mathcal{E}$ is isomorphic to a quotient of an $\mathcal{O}_Y$-module of the form $g^*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{K}^{\otimes -m}$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Z$-module of finite type (depending on m).*

[Then it is shown that if $f^*(\mathcal{E}) \to \mathcal{L}$ induces an immersion $X \to \mathbf{P}(\mathcal{E})$, and there is a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$ and a surjection $g^*(\mathcal{F}) \to \mathcal{E} \otimes \mathcal{K}^{\otimes m}$, then $\mathcal{L} \otimes \mathcal{K}^{\otimes(m+1)}$ is very ample for $X \to Z$.]

Proposition (4.4.11). — *Let $X'' = X \coprod X'$ be a prescheme disjoint union, $f''\colon X'' \to Y$ a morphism restricting to morphisms $f\colon X \to Y$, $f'\colon X' \to Y$. Let $\mathcal{L}$, $\mathcal{L}'$ be invertible $\mathcal{O}_X$, $\mathcal{O}_{X'}$-modules, $\mathcal{L}''$ the invertible $\mathcal{O}_{X''}$-module restricting to $\mathcal{L}$, $\mathcal{L}'$. Then $\mathcal{L}''$ is very ample iff $\mathcal{L}$ and $\mathcal{L}'$ are very ample.*

4.5. **Ample sheaves.**

(4.5.1). Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $S = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ is a positively graded subring of $\Gamma_*(\mathcal{L})$ (0, 5.4.6). Let $p\colon X \to \mathrm{Spec}(\mathbb{Z})$ be the structure morphism. We have a canonical graded $\mathcal{O}_X$-algebra homomorphism $\varepsilon\colon p^*(\widetilde{S}) \to \mathbf{S}(\mathcal{L}) = \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ by adjointness of $p_* = \Gamma(X, -)$ and p^*. Then (3.7.1) provides a canonical morphism $G(\varepsilon) \to \mathrm{Proj}(S)$.

When $\mathcal{L}$ is understood, define $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for any $\mathcal{O}_X$-module $\mathcal{F}$.

Theorem (4.5.2). — *Let X be a quasi-compact scheme or a prescheme with Noetherian underlying space, and $\mathcal{L}$, S as above. The following are equivalent:*

(a) The sets X_f for homogeneous $f \in S_+$ form a base of the topology on X.

(a′) Those X_f which are affine cover X.

(b) The canonical morphism $G(\varepsilon) \to \mathrm{Proj}(S)$ is defined on all of X and is a dominant open immersion.

(b′) $G(\varepsilon) \to \mathrm{Proj}(S)$ is defined on all of X and is a homeomorphism of X onto a subspace of $\mathrm{Proj}(S)$.

(c) For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, let $\mathcal{F}_n$ be the submodule of $\mathcal{F}(n)$ generated by its global sections on X. Then $\mathcal{F}$ is the sum of its sub-$\mathcal{O}_X$-modules of the form $\mathcal{F}_n(-n)$, as n ranges over all positive integers.

(c′) Property (c) holds for quasi-coherent sheaves of ideals in $\mathcal{O}_X$.

Moreover, given homogeneous elements (f_α) in S_+ such that X_{f_α} is affine, the canonical morphism $X \to \mathrm{Proj}(S)$ restricts to an isomorphism $\bigcup_\alpha X_{f_\alpha} \cong \bigcup_\alpha D_+(f_\alpha) \subseteq \mathrm{Proj}(S)$.

[Proof: The preimage of $D_+(f)$ is X_f, and $G(\varepsilon)$ is the union of these. On any affine $U \subseteq X$ such that $\mathcal{L}|U \cong \mathcal{O}_U$ is trivial we have $X_f \cap U \cong U_{f'}$ for a section f' of $\mathcal{O}_U$ corresponding to f. So (b) ⇒ (b′) ⇒ (a) ⇒ (a′). By (I, 9.3.1–2) and (3.8.2), (a′) implies the “moreover,” which together with (a′) implies (b). (I, 9.3.1) gives (a) ⇒ (c), clearly (c) ⇒ (c′), and (c) ⇒ (a) by taking for any open $U \subseteq X$ an ideal $\mathcal{J}$ such that $V(\mathcal{J})$ is the complement of U.]

Condition (b) implies that X is a *scheme*.

The proof also shows that those X_f which are affine form a base of the topology.

Definition (4.5.3). — An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is called *ample* if X is a quasi-compact scheme and the conditions in (4.5.2) hold.

By (a), if $\mathcal{L}$ is ample, then so is $\mathcal{L}|U$ for any quasi-compact open subset $U \subseteq X$.

Corollary (4.5.4). — *If $\mathcal{L}$ is ample, $Z \subseteq X$ is a finite subset, and U is a neighborhood of Z, there exists n and $f \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_f is an affine neighborhood of Z contained in U.*

[This uses a lemma from commutative algebra, that if $\mathfrak{p}_i$ are finitely many homogeneous prime ideals, not containing an ideal $I \subseteq S$, then there is a homogeneous element of I not contained in the union of the ideals $\mathfrak{p}_i$.]

Proposition (4.5.5). — *Let X be a quasi-compact scheme or a prescheme with Noetherian underlying space. The conditions in (4.5.2) are also equivalent to the following:*

(d) For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists n_0 such that $\mathcal{F}(n)$ is generated by its global sections for all $n \geq n_0$.

(d′) Every such $\mathcal{F}$ is isomorphic to a quotient of an $\mathcal{O}_X$-module of the form $\mathcal{L}^{\otimes(-n)} \otimes \mathcal{O}_X^k$.

(d″) Property (d′) holds for quasi-coherent ideal sheaves of finite type in $\mathcal{O}_X$.

[(c′) ⇒ (d) ⇒ (d′) ⇒ (d″) are straightforward. (d″) ⇒ (a) uses (9.4.9)]

Proposition (4.5.6). — *Let X be a quasi-compact scheme, $\mathcal{L}$ an inverible $\mathcal{O}_X$-module.*

(i) For $n > 0$, $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes n}$ is ample.

(ii) Let $\mathcal{L}'$ be invertible and assume that for every $x \in X$ there exists $n > 0$ and $s \in \Gamma(X, \mathcal{L}'^{\otimes n})$ such that $s(x) \neq 0$. Then $\mathcal{L}$ ample implies $\mathcal{L} \otimes \mathcal{L}'$ ample.

Corollary (4.5.7). — *The tensor product of ample $\mathcal{O}_X$-modules is ample.*

Corollary (4.5.8). — *If $\mathcal{L}$ is ample, $\mathcal{L}'$ invertible, there exists $n_0 > 0$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is ample for all $n \geq n_0$.*

Remark (4.5.9). — In the [Picard group] $P \cong H^1(X, \mathcal{O}_X^*)$ of invertible sheaves on X, the ample sheaves form a subset P^+ such that

$$P_+ + P_+ \subseteq P_+, \quad P_+ - P_+ = P.$$

Hence P is a quasi-ordered abelian group with $P_+ \cup \{0\}$ its *positive* cone.

Proposition (4.5.10). — *Let Y be affine, $q\colon X \to Y$ quasi-compact and separated, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

(i) If $\mathcal{L}$ is very ample for q, then $\mathcal{L}$ is ample.

(ii) Suppose q is of finite type. *Then $\mathcal{L}$ is ample iff the following equivalent conditions hold:*

(e) There exists $n_0 > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample for all $n \geq n_0$.

(e′) $\mathcal{L}^{\otimes n}$ is very ample for some $n > 0$.

(4.5.10.1). *Proof of Lemma (4.4.10.1).* — Let $\mathcal{E}(n) = \mathcal{E} \otimes \mathcal{K}^{\otimes n}$. For large n, we want to find a quasi-coherent subsheaf $\mathcal{F} \subseteq g_*(\mathcal{E}(n))$ of finite type such that the canonical map $g^*(\mathcal{F}) \to \mathcal{E}(n)$ is surjective. By quasi-compactness and (9.4.7), we can reduce to the case that Z is affine. Then (4.5.10, (i)) and (4.5.5, (d)) give the result.

Corollary (4.5.11). — *If Y is affine, $q\colon X \to Y$ separated and of finite type, $\mathcal{L}$ ample, $\mathcal{L}'$ invertible, there exists n_0 such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is very ample for q, for all $n \geq n_0$.*

Remark (4.5.12). — It is not known whether $\mathcal{L}^{\otimes n}$ very ample implies the same for $\mathcal{L}^{\otimes(n+1)}$.

Proposition (4.5.13). — *Let X be quasi-compact, $Z \subseteq X$ a closed sub-prescheme defined by a* nilpotent *sheaf of ideals, $j\colon Z \hookrightarrow X$ the inclusion. Then $\mathcal{L}$ is ample iff $\mathcal{L}' = j^*(\mathcal{L})$ is ample.*

[The proof relies on the following lemma, which in turn is proved using sheaf cohomology.]

Lemma (4.5.13.1). — *In (4.5.13), suppose further that $\mathcal{J}^2 = 0$, and let $g \in \Gamma(Z, \mathcal{L}'^{\otimes n})$ be such that Z_g is affine. Then there exists $m > 0$ such that $g^{\otimes m} = j^*(f)$ for a global section $f \in \Gamma(X, \mathcal{L}^{\otimes mn})$.*

Corollary (4.5.14). — *Let X be a Noetherian scheme, $j\colon X_{\mathrm{red}} \to X$ the inclusion. Then $\mathcal{L}$ is ample if and only if $j^*\mathcal{L}$ is ample.*

4.6. **Relatively ample sheaves.**

Definition (4.6.1). — Let $f\colon X \to Y$ be a quasi-compact morphism, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample relative to* f, or f*-ample*, or *ample relative to* Y (when f is understood) if there exists an open affine cover (U_α) of Y such that for every α, $\mathcal{L}|f^{-1}(U_\alpha)$ is ample.

Note that the existence of a relatively ample sheaf entails that f must be separated (4.5.3).

Proposition (4.6.2). — *Let $f\colon X \to Y$ be quasi-compact. If $\mathcal{L}$ is very ample for f, then $\mathcal{L}$ is ample relative to f.*

Proposition (4.6.3). — *Let $f\colon X \to Y$ be quasi-compact, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and put $\mathcal{S} = \bigoplus_{n\geq 0} f_*(\mathcal{L}^{\otimes n})$, a graded $\mathcal{O}_Y$-algebra. The following are equivalent:*

(a) $\mathcal{L}$ is f-ample.

(b) $\mathcal{S}$ is quasi-coherent and the canonical homomorphism $\sigma\colon f^*(\mathcal{S}) \to$ $\mathbf{S}(\mathcal{L})$ *(0, 4.4.3) induces an everywhere-defined, dominant open immersion $r_{\mathcal{L},\sigma}\colon X \hookrightarrow P =$* $\mathrm{Proj}(\mathcal{S})$.

(b′) f is separated, and the morphism $r_{\mathcal{L},\sigma}$ is everywhere defined and is a homeomorphism of X onto a subspace of $\mathrm{Proj}(\mathcal{S})$.

Moreover, when these conditions hold, the canonical homomorphism $r^*_{\mathcal{L},\sigma}(\mathcal{O}_P(n)) \to \mathcal{L}^{\otimes n}$ (3.7.9.1) is an isomorphism. Furthermore, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, if we put $\mathcal{M} = \bigoplus_{n\geq 0} f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$, then $r^*_{\mathcal{L},\sigma}(\widetilde{\mathcal{M}}) \to \mathcal{F}$ (3.7.9.2) is an isomorphism.

Corollary (4.6.4). — *Let (U_α) be an open affine covering of Y. Then $\mathcal{L}$ is ample relative to f if and only if $\mathcal{L}|f^{-1}(U_\alpha)$ is ample relative to U_α, for all α.*

Corollary (4.6.5). — *Let $\mathcal{K}$ be an invertible $\mathcal{O}_Y$-module. Then $\mathcal{L}$ is f-ample iff $\mathcal{L} \otimes f^*(\mathcal{K})$ is.*

Corollary (4.6.6). — *Suppose Y affine. Then $\mathcal{L}$ is Y-ample iff it is ample.*

Corollary (4.6.7). — *Let $f\colon X \to Y$ be a quasi-compact morphism. Suppose there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ and a morphism $g\colon X \to P = \mathrm{Proj}(\mathcal{E})$ which is a homeomorphism of X onto a subspace of P. Then $\mathcal{L} = g^*(\mathcal{O}_P(1))$ is f-ample.*

Proposition (4.6.8). — *Let X be a quasi-compact scheme or a prescheme with Noetherian underlying space, $f\colon X \to Y$ a quasi-compact, separated morphism. An invertible $\mathcal{O}_X$-module $\mathcal{L}$ is f-ample if and only if the following equivalent conditions hold:*

(c) For every $\mathcal{O}_X$-module $\mathcal{F}$ of finite type, there exists $n_0 > 0$ such that the canonical homomorphism $\sigma\colon f^*(f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is surjective for all $n \geq n_0$.

(c′) Property (c) holds for all $\mathcal{F} = \mathcal{J} \subseteq \mathcal{O}_X$ a quasi-coherent ideal sheaf of finite type.

Proposition (4.6.9). — *Let $f\colon X \to Y$ be a quasi-compact morphism, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

(i) Let $n > 0$. Then $\mathcal{L}$ is f-ample iff $\mathcal{L}^{\otimes n}$ is.

(ii) Let $\mathcal{L}'$ be an invertible $\mathcal{O}_X$-module such that $\sigma\colon f^*(f_*(\mathcal{L}'^{\otimes n})) \to \mathcal{L}'^{\otimes n}$ for some $n > 0$. Then if $\mathcal{L}$ is f-ample, so is $\mathcal{L} \otimes \mathcal{L}'$.

Corollary (4.6.10). — *The tensor product of f-ample $\mathcal{O}_X$-module is f-ample.*

Proposition (4.6.11). — *Let Y be quasi-compact, $f\colon X \to Y$ a morphism of finite type, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample iff the following equivalent conditions hold:*

(d) There exists $n_0 > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample for f, for all $n \geq n_0$.

(d′) There exists $n > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample for f.

Corollary (4.6.12). — *Let Y be quasi-compact, $f\colon X \to Y$ of finite type, $\mathcal{L}$, $\mathcal{L}'$ invertible $\mathcal{O}_X$-modules. If $\mathcal{L}$ is f-ample, there exists n_0 such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is very ample for f, for all $n \geq n_0$.*

Proposition (4.6.13). — *(i) Every invertible $\mathcal{O}_Y$-module $\mathcal{L}$ is ample relative to the identity map $1_Y\colon Y \to Y$.*

(i′) Let $f\colon X \to Y$ be quasi-compact, $j\colon X' \to X$ a quasi-compact morphism which is a homeomorphism of X' onto a subspace of X. If $\mathcal{L}$ is f-ample, then $j^*\mathcal{L}$ is ample relative to $f \circ j$.

(ii) Let Z be quasi-compact, $f\colon X \to Y$, $g\colon Y \to Z$ quasi-compact morphisms, $\mathcal{L}$ f-ample, $\mathcal{K}$ g-ample. Then there exists $n_0 > 0$ such that $\mathcal{L} \otimes f^*(\mathcal{K}^{\otimes n})$ is ample relative to $g \circ f$, for all $n \geq n_0$.

(iii) Let $f\colon X \to Y$ be quasi-compact $g\colon Y' \to Y$ any morphism. If $\mathcal{L}$ is f-ample, then $\mathcal{L} \otimes_Y \mathcal{O}_{Y'}$ is ample relative to $f_{(Y')}$.

(iv) Let $f_i\colon X_i \to Y_i$ $(i = 1, 2)$ be quasi-compact S-morphisms. If $\mathcal{L}_i$ is ample relative to f_i, then $\mathcal{L}_1 \otimes_S \mathcal{L}_2$ is ample relative to $f_1 \times_S f_2$.

(v) Let $f\colon X \to Y$, $g\colon Y \to Z$, be such that $g \circ f$ is quasi-compact. Assume that g is separated, or that X has locally Noetherian underlying space. If $\mathcal{L}$ is ample relative to $g \circ f$, then $\mathcal{L}$ is f-ample.

(vi) Let $f\colon X \to Y$ be quasi-compact, $j\colon X_{\mathrm{red}} \hookrightarrow X$ the inclusion. If $\mathcal{L}$ is f-ample, then $j^*\mathcal{L}$ is ample relative to f_{red}.

[Assertions (i), (i′), (iii) and (iv) imply the rest; (i) is trivial from (4.4.10, (i)) and (4.6.2). The others are proved using the following lemma.]

Lemma (4.6.13.1). — *(i) Let $u\colon Z \to S$ be a morphism $\mathcal{L}$ an invertible $\mathcal{O}_S$-module, $\mathcal{L}' = u^*(\mathcal{L})$, $s \in \Gamma(S, \mathcal{L})$, $s' = u^*(s)$. Then $Z_{s'} = u^{-1}(S_s)$.*

(ii) Let Z, Z' be S-preschemes, $T = Z \times_S Z'$, p, p' the projections, $\mathcal{L}$ (resp. $\mathcal{L}$) and invertible $\mathcal{O}_Z$-module (resp. $\mathcal{O}_{Z'}$-module), $t \in \Gamma(Z, \mathcal{L})$, $t' \in \Gamma(Z', \mathcal{L}')$, $s = p^*(t)$, $s' = p'^*(t')$. Then $T_{s \otimes s'} = Z_t \times_S Z'_{t'}$.

Remark (4.6.14). — In (ii) it need not be the case that $\mathcal{L} \otimes f^*(\mathcal{K})$ is ample relative to $g \circ f$. Were this so, one could take $\mathcal{L}' = \mathcal{L} \otimes f^*(\mathcal{K}^{-1})$ in the place of $\mathcal{L}$ and conclude that $\mathcal{L}$ is ample relative to $g \circ f$, for any invertible $\mathcal{O}_X$-module $\mathcal{L}$, which is clearly false (suppose g were the identity!).

Proposition (4.6.15). — *Let $f\colon X \to Y$ be quasi-compact, $\mathcal{J} \subseteq \mathcal{O}_X$ a locally nilpotent quasi-coherent ideal sheaf, $j\colon Z = V(\mathcal{J}) \hookrightarrow X$ the inclusion of the closed subscheme defined by $\mathcal{J}$. Then $\mathcal{L}$ is ample for f if and only if $j^*(\mathcal{L})$ is ample for $f \circ j$.*

Corollary (4.6.16). — *Let X be locally Noetherian, $f\colon X \to Y$ quasi-compact, $j\colon X_{\mathrm{red}} \hookrightarrow X$ the inclusion. Then $\mathcal{L}$ is ample for f if and only if $j^*(\mathcal{L})$ is ample for f_{red}.*

Proposition (4.6.17). — *With the notation and hypotheses of (4.4.11), $\mathcal{L}''$ is ample relative to f'' iff $\mathcal{L}$ is ample relative to f and $\mathcal{L}'$ is ample relative to f'.*

Proposition (4.6.18). — *Let Y be quasi-compact, $\mathcal{S}$ a graded quasi-coherent $\mathcal{O}_Y$-algebra of finite type, $X = \mathrm{Proj}(\mathcal{S})$, $f\colon X \to Y$ the structure morphism. Then f is of finite type, and $\mathcal{O}_X(d)$ is invertible and f-ample for some $d > 0$.*