Synopsis of material from EGA Chapter II, §3

3. Homogeneous spectrum of a sheaf of graded algebras

3.1. **Homogeneous spectrum of a graded quasi-coherent $\mathcal{O}_Y$ algebra.**

(3.1.1). Let Y be a prescheme. A sheaf of graded $\mathcal{O}_Y$ algebras $\mathcal{S} = \bigoplus_n \mathcal{S}_n$ is quasi-coherent iff each $\mathcal{S}_n$ is; similarly for a graded $\mathcal{S}$ module $\mathcal{M}$. The notations $\mathcal{S}^{(d)}$, $\mathcal{M}(n)$, etc., are used analogously to those for graded algebras and modules [see (2.1.1)].

Let $U = \operatorname{Spec}(A) \subseteq Y$ be an open affine. Then $\mathcal{S}|U = \widetilde{S}$, where $S = \Gamma(U, \mathcal{S})$ is a graded A algebra. Set $X_U = \operatorname{Proj}(S)$. Given another affine $U' = \operatorname{Spec}(A') \subseteq U$, we have a ring homomorphism $A \to A'$ corresponding to $U' \hookrightarrow U$, and the restriction homomorphism $S \to S' = \Gamma(U', \mathcal{S})$ is just the induced map $S \to S' = S \otimes_A A'$ (I, 1.6.4). Hence $X_{U'} = X_U \times_U U'$ by (2.8.10), *i.e.*, $X_{U'} = f_U^{-1}(U')$, where $f_U \colon X_U \to U$ is the stucture morphism. Let $\rho_{U',U} \colon X_{U'} \to X_U$ be the open immersion thus defined. Given $U'' \subseteq U' \subseteq U$, we have $\rho_{U'',U} = \rho_{U',U} \circ \rho_{U'',U'}$.

Proposition (3.1.2). — *Given a quasi-coherent sheaf of positively graded $\mathcal{O}_Y$ algebras $\mathcal{S}$, there is a prescheme $f \colon X \to Y$ over Y, unique up to canonical isomorphism, such that for every open affine $U \subseteq Y$, $f^{-1}(U)$ is identified with X_U, in such a way that for every $U' \subseteq U$, the inclusion $f^{-1}(U') \subseteq f^{-1}(U)$ is identified with $\rho_{U',U}$.*

(3.1.3). The prescheme X in (3.1.2) is the *homogeneous spectrum of* $\mathcal{S}$, denoted $\operatorname{Proj}(\mathcal{S})$. X is separated over Y by (2.4.2) and (I, 5.5.5), and if $\mathcal{S}$ is an $\mathcal{O}_Y$ algebra of finite type (I, 9.6.2), then X is of finite type over Y. For any open $U \subseteq Y$, we clearly have $f^{-1}(U) \cong \operatorname{Proj}(\mathcal{S}|U)$.

Proposition (3.1.4). — *Let $f \in \Gamma(Y, \mathcal{S}_d)$, $d > 0$. There is an open subset $X_f \subseteq X$ such that $X_f \cap X_U$ is the basic open set $D_+(f|U)$ of X_U for each affine U. In particular, $X_f \cong \operatorname{Spec}(\mathcal{S}^{(d)}/(f-1)\mathcal{S}^{(d)})$ is affine over Y.*

We call X_f the *non-vanishing locus* of f.

Corollary (3.1.5). — $X_{fg} = X_f \cap X_g$.

Corollary (3.1.6). — *If (f_α) is a family of homogeneous sections of $\mathcal{S}$, and if the sheaf of ideals in $\mathcal{S}$ that they generate contains $\mathcal{S}_n$ for all sufficiently large n, then the open sets X_{f_α} cover X.*

Corollary (3.1.7). — *If $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$ algebras, then* $\operatorname{Proj}(\mathcal{A}[t]) = \operatorname{Spec}(\mathcal{A})$. *In particular,* $\operatorname{Proj}(\mathcal{O}_Y[t]) = Y$.

Proposition (3.1.8). — *(i)* $\operatorname{Proj}(\mathcal{S}) \cong \operatorname{Proj}(\mathcal{S}^{(d)})$ *as a Y-scheme [see (2.4.7, (i))].*

(ii) Let $\mathcal{S}' = \mathcal{O}_Y \oplus \bigoplus_{n>0} \mathcal{S}_n$. Then $\operatorname{Proj}(\mathcal{S}) \cong \operatorname{Proj}(\mathcal{S}')$ *as a Y-scheme [see (2.4.8)].*

(iii) Let $\mathcal{L}$ be an invertible sheaf on Y, and let $\mathcal{S}_{(\mathcal{L})} = \bigoplus_n (\mathcal{S}_n \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes n})$. Then there is a canonical isomorphism of Y-schemes $\operatorname{Proj}(\mathcal{S}) \cong \operatorname{Proj}(\mathcal{S}_{(\mathcal{L})})$.

(3.1.9). By (0, 4.1.3) and (I, 1.3.14), $\mathcal{S}_1$ generates $\mathcal{S}$ iff $\Gamma(U_\alpha, \mathcal{S}_1)$ generates $\Gamma(U_\alpha, \mathcal{S})$, for all U_α in an affine covering; and if so, this holds for every affine $U \subseteq Y$.

Proposition (3.1.10). — *Suppose Y has a finite affine open covering (U_i) such that each $\Gamma(U_i, \mathcal{S})$ is of finite type over $\Gamma(U_i, \mathcal{O}_Y)$. Then for some d, $\mathcal{S}^{(d)}$ is generated by $\mathcal{S}_d$, and $\mathcal{S}_d$ is an $\mathcal{O}_Y$ module of finite type.*

Corollary (3.1.11). — *Under the hypotheses of (3.1.10),* $\operatorname{Proj}(\mathcal{S}) \cong \operatorname{Proj}(\mathcal{S}')$*, where $\mathcal{S}'$ is generated as an $\mathcal{O}_Y$ algebra by $\mathcal{S}'_1$, which is a finitely generated $\mathcal{O}_Y$ module.*

(3.1.12). Let $\mathcal{N}$ be the nilradical of $\mathcal{S}$. It's quasi-coherent by (I, 5.5.1). Put $\mathcal{N}_+ = \mathcal{N} \cap \mathcal{S}_+$, a graded $\mathcal{S}_0$ module by (2.1.10). We call $\mathcal{S}$ *essentially reduced* if $\mathcal{N}_+ = 0$, which is equivalent to $\mathcal{S}_y$ being essentially reduced [see (2.1.10)] for all $y \in Y$. We call $\mathcal{S}$ *integral* if $\mathcal{S}_y$ is an integral domain with $(\mathcal{S}_y)_+ \neq 0$ for all $y \in Y$.

Proposition (3.1.13). — *If* $X = \operatorname{Proj}(\mathcal{S})$*, then* $X_{\mathrm{red}} \cong \operatorname{Proj}(\mathcal{S}/\mathcal{N}_+)$*. In particular, X is reduced if $\mathcal{S}$ is essentially reduced [seee (2.4.4, (i))].*

Proposition (3.1.14). — *Let Y be an integral prescheme, $\mathcal{S}$ a graded quasi-coherent $\mathcal{O}_Y$ algebra such that $\mathcal{S}_0 = \mathcal{O}_Y$.*
(i) If $\mathcal{S}$ is integral (3.1.12), then $X = \operatorname{Proj}(\mathcal{S})$ *is integral, and the structure morphism $\phi\colon X \to Y$ is dominant.*
(ii) Conversely, if $\mathcal{S}$ is essentially reduced, X is integral, and ϕ is dominant, then $\mathcal{S}$ is integral.

3.2. **Sheaf on** $\operatorname{Proj}(\mathcal{S})$ **associated to a graded** $\mathcal{S}$ **module.**

(3.2.1). Let $\mathcal{S}$ be a quasi-coherent sheaf of graded $\mathcal{O}_Y$ modules, $\mathcal{M}$ a quasi-coherent sheaf of graded $\mathcal{S}$ modules (quasi-coherent as an $\mathcal{S}$ module sheaf equivalently as an $\mathcal{O}_Y$ module sheaf). Keeping the notation of (3.1.1), let $\widetilde{\mathcal{M}_U}$ be the sheaf on X_U associated to $\Gamma(U, \mathcal{M})$ (2.5.3). If $U' \subseteq U$, then $\Gamma(U', \mathcal{M}) = \Gamma(U, \mathcal{M}) \otimes_A A'$, hence $\widetilde{\mathcal{M}_{U'}} = \rho^*_{U',U} \widetilde{\mathcal{M}_U} = \widetilde{\mathcal{M}_U} | X_{U'}$.

Proposition (3.2.2). — *There is a unique quasi-coherent $\mathcal{O}_X$ module $\widetilde{\mathcal{M}}$ such that $\widetilde{\mathcal{M}} | X_U = \widetilde{\mathcal{M}_U}$ for all open affines $U \subseteq Y$.*

Proposition (3.2.3). — *Let $f \in \Gamma(Y, \mathcal{S}_d)$, $d > 0$. The isomorphism* $X_f \cong \operatorname{Spec}(\mathcal{S}^{(d)}/(f-1)\mathcal{S}^{(d)})$ *identifies $\widetilde{\mathcal{M}} | X_f$ with the sheaf associated to the $\mathcal{S}^{(d)}/(f-1)\mathcal{S}^{(d)}$ module $\mathcal{M}^{(d)}/(f-1)\mathcal{M}^{(d)}$ [see (2.8.12)].*

Proposition (3.2.4). — *$\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is an exact, covariant functor, which preserves direct sums and direct limits.*

In particular, if $\mathcal{J} \subseteq \mathcal{S}$ is a homogeneous ideal sheaf, then $\widetilde{\mathcal{J}}$ is a sheaf of ideals in $\mathcal{O}_X$. If $\mathcal{I}$ is a sheaf of ideals in $\mathcal{O}_Y$, then $(\mathcal{I}\mathcal{M})^{\sim} = \mathcal{I} \cdot \widetilde{\mathcal{M}}$.

Proposition (3.2.5). — *Let $f \in \mathcal{S}_d$. The restriction of $\mathcal{S}(nd)^{\sim}$ to X_f is isomorphic to $\mathcal{O}_{X_f}$ [with generating section f^n]. In particular, if $\mathcal{S}_1$ generates $\mathcal{S}$, then each $\mathcal{S}(n)^{\sim}$ is invertible [see (2.5.7–9)].*

As before, we define

$$\mathcal{O}_X(n) = \mathcal{S}(n)^{\sim}, \tag{3.2.5.1}$$

$$\mathcal{F}_X(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n). \tag{3.2.5.2}$$

Proposition (3.2.6). — *There are canonical, functorial homomorphisms*

$$\lambda\colon \widetilde{\mathcal{M}} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{N}} \to (\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N})^{\sim}, \tag{3.2.6.1}$$

$$\mu\colon \mathcal{H}om_{\mathcal{S}}(\mathcal{M}, \mathcal{N})^{\sim} \to \mathcal{H}om_{\mathcal{O}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}). \tag{3.2.6.2}$$

If $\mathcal{S}_1$ generates $\mathcal{S}$, then λ is an isomorphism, and if in addition $\mathcal{M}$ is finitely presented, then μ is an isomorphism [see (2.5.11-13)].

Corollary (3.2.7). — *[see (2.5.14)] If $\mathcal{S}_1$ generates $\mathcal{S}$, then for all $m, n \in \mathbb{Z}$,*

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n) \tag{3.2.7.1}$$

$$\mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}. \tag{3.2.7.2}$$

Corollary (3.2.8). — *[see (2.5.15)] If $\mathcal{S}_1$ generates $\mathcal{S}$, then $(\mathcal{M}(n))^{\sim} = \widetilde{\mathcal{M}}(n)$.*

Remarks (3.2.9). — (i) If $\mathcal{S} = \mathcal{A}[t]$ as in (3.1.7), then $\mathcal{O}_X(n) \cong \mathcal{O}_X$ for all n. If $\mathcal{N}$ is a quasi-coherent sheaf of $\mathcal{A}$ modules, $\mathcal{M} = \mathcal{N} \otimes_{\mathcal{A}} \mathcal{A}[t]$, then $\widetilde{\mathcal{M}}$ is the sheaf on $X \cong \operatorname{Spec}(\mathcal{A})$ associated to $\mathcal{N}$ as in (1.4.3).

(ii) If $\mathcal{S}'_0 = \mathcal{O}_Y$, $\mathcal{S}'_n = \mathcal{S}_n$ for $n > 0$, then the canonical isomorphism $X \cong X'$ identifies $\mathcal{O}_X(n)$ with $\mathcal{O}_{X'}(n)$. If $X^{(d)} = \operatorname{Proj}(\mathcal{S}^{(d)})$, the canonical isomorphism $X \cong X^{(d)}$ identifies $\mathcal{O}_{X^{(d)}}(n)$ with $\mathcal{O}_X(nd)$ [see (2.5.16)].

Proposition (3.2.10). — *Let $\mathcal{L}$ be an invertible sheaf on Y. The canonical isomorphism $X_{(\mathcal{L})} = \operatorname{Proj}(\mathcal{S}_{(\mathcal{L})}) \cong X = \operatorname{Proj}(S)$ in (3.1.8, (iii)) identifies $\mathcal{O}_{X_{(\mathcal{L})}}(n)$ with $\mathcal{O}_X(n) \otimes_Y \mathcal{L}^{\otimes n}$.*

3.3. **Graded $\mathcal{S}$ module associated with a sheaf on $\operatorname{Proj}(\mathcal{S})$.**

In this section we assume that $\mathcal{S}_1$ generates $\mathcal{S}$. Recall that by (3.1.8 (i)), this is no essential restriction when the finiteness conditions in (3.1.10) hold.

(3.3.1). Let $p\colon X = \operatorname{Proj}(\mathcal{S}) \to Y$ be the structure morphism. For any sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$, put

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{F}(n)). \tag{3.3.1.1}$$

In particular,

$$\Gamma_*(\mathcal{O}) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{O}(n)). \tag{3.3.1.2}$$

The canonical homomorphism (0, 4.2.2) $p_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} p_*(\mathcal{G}) \to p_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$ makes $\Gamma_*(\mathcal{O})$ a graded $\mathcal{O}_Y$ algebra, and $\Gamma_*(\mathcal{F})$ a graded $\Gamma_*(\mathcal{O})$ module. The functor $\Gamma_*(-)$ is left exact; in particular, if $\mathcal{J} \subseteq \mathcal{O}_X$ is an ideal sheaf, then $\Gamma_*(\mathcal{J})$ is a homogeneous ideal sheaf in $\Gamma_*(\mathcal{O})$.

(3.3.2). As in (2.6.2), for any graded $\mathcal{S}$ module $\mathcal{M}$, there is a canonical homomorphism of graded sheaves

$$\alpha\colon \mathcal{M} \to \Gamma_*(\widetilde{\mathcal{M}}). \tag{3.3.2.3}$$

In particular, $\alpha\colon \mathcal{S} \to \Gamma_*(\mathcal{O})$ is a homomorphism of sheaves of graded algebras, which makes $\Gamma_*(\widetilde{\mathcal{M}})$ a graded $\mathcal{S}$ module, and (3.3.2.3) an $\mathcal{S}$ module homomorphism.

Note that α_n induces $p^*(\mathcal{M}_n) \to \widetilde{\mathcal{M}}(n)$; this is the sheaf homomorphism associated by (3.2.4) to the canonical graded $\mathcal{O}_Y$ module homomorphism $\mathcal{M}_n \otimes_{\mathcal{O}_Y} \mathcal{S} \to \mathcal{M}(n)$.

Proposition (3.3.3). — *Given $f \in \Gamma(X, \mathcal{S}_d)$, $d > 0$, X_f is the non-vanishing locus of the section $\alpha_d(f)$ of the invertible sheaf $\mathcal{O}_X(d)$.*

(3.3.4). Suppose now that for every quasi-coherent $\mathcal{F}$ on X, the sheaves $p_*(\mathcal{F})$ and hence $\Gamma_*(\mathcal{F})$ are quasi-coherent on Y. In particular, this holds if X is of finite type over Y (I, 9.2.2). Then $\Gamma_*(\mathcal{F})^{\sim}$ is defined and quasi-coherent on X. As in (2.6.4), there is a canonical homomorphism

$$\beta\colon \Gamma_*(\mathcal{F})^{\sim} \to \mathcal{F}. \tag{3.3.4.1}$$

Proposition (3.3.5). — *Let $\mathcal{M}$ be a quasi-coherent graded $\mathcal{S}$ module, $\mathcal{F}$ a quasi-coherent sheaf on $\mathcal{O}_X$. Then each of the following maps in the identity [see (2.6.5)]:*

$$\widetilde{\mathcal{M}} \xrightarrow{\tilde{\alpha}} \Gamma_*(\widetilde{\mathcal{M}})^{\sim} \xrightarrow{\beta} \widetilde{\mathcal{M}}, \tag{3.3.5.1}$$

$$\Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*(\Gamma_*(\mathcal{F})^{\sim}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F}). \tag{3.3.5.2}$$

3.4. **Finiteness conditions.**

Proposition (3.4.1). — *Let $\mathcal{S}$ be a quasi-coherent sheaf of graded $\mathcal{O}_Y$ algebras, generated by $\mathcal{S}_1$, and suppose further that $\mathcal{S}_1$ is an $\mathcal{O}_Y$ module of finite type. Then $X = \operatorname{Proj}(\mathcal{S})$ is of finite type over Y [see (2.7.1, (ii))].*

(3.4.2). Consider two conditions on a graded $\mathcal{S}$ module $\mathcal{M}$:
(TF) There exists n such that $\bigoplus_{k \geq n} \mathcal{M}_k$ is a sheaf of $\mathcal{S}$ modules of finite type;
(TN) There exists n such that $\mathcal{M}_k = 0$ for $k \geq n$.
The terminology of (2.7.2) will be used in this context also.

Proposition (3.4.3). — *[see (2.7.3)] Assume that $\mathcal{S}_1$ is of finite type and generates $\mathcal{S}$.*
(i) If $\mathcal{M}$ satisfies (TF), then $\widetilde{\mathcal{M}}$ is of finite type.
(ii) If $\mathcal{M}$ satisfies (TF), then $\widetilde{\mathcal{M}} = 0$ if and only if M satisfies (TN).

Theorem (3.4.4). — *Assume that $\mathcal{S}_1$ is of finite type and generates $\mathcal{S}$. For every quasi-coherent sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$, the canonical homomorphism β in (3.3.4) is an isomorphism [see (2.7.5)].*

Corollary (3.4.5). — *Under the hypotheses of (3.4.4), every quasi-coherent $\mathcal{O}_X$ module $\mathcal{F}$ is of the form $\widetilde{\mathcal{M}}$ for some graded $\mathcal{S}$ module $\mathcal{M}$ [see (2.7.7)]. If $\mathcal{F}$ is of finite type, and if Y is quasi-compact and separated, or if its underlying space is Noetherian, then $\mathcal{M}$ can be taken to be of finite type [see (2.7.8)—the hypotheses on Y serve to imply that X is quasi-compact, by (3.4.1)].*

Corollary (3.4.6). — *Under the hypotheses of (3.4.4), suppose further that Y is quasi-compact, and $\mathcal{F}$ is of finite type. Then the canonical homomorphism $\sigma\colon p^*(p_*(\mathcal{F}(n))) \to \mathcal{F}(n)$ is surjective for all sufficiently large n.*

Remarks (3.4.7). — For any morphism $p\colon X \to Y$ of ringed spaces, and $\mathcal{O}_X$ module $\mathcal{F}$, the surjectivity of $\sigma\colon p^*(p_*(\mathcal{F})) \to \mathcal{F}$ amounts to the following: for every $x \in X$ and every section s of $\mathcal{F}$ on a neighborhood V of x, there is a neighborhood U of $p(x)$ in Y, a neighborhood $W \subseteq V \cap p^{-1}(U)$ of x, and finitely many sections $t_i \in \mathcal{F}(p^{-1}(U))$ and $a_i \in \mathcal{O}_X(W)$, such that

$$s|W = \sum_i a_i(t_i|W).$$

If Y is an affine scheme, and $p_*(\mathcal{F})$ is quasi-coherent, this is equivalent to $\mathcal{F}$ being generated by its global sections on X. Hence for any morphism $p\colon X \to Y$ of preschemes, and any quasi-coherent $\mathcal{O}_X$ module $\mathcal{F}$ such that $p_*(\mathcal{F})$ is quasi-coherent, the following are equivalent:

(a) $\sigma\colon p^*(p_*(\mathcal{F})) \to \mathcal{F}$ is surjective;

(b) there exists a quasi-coherent $\mathcal{O}_Y$ module $\mathcal{G}$ such that $p^*(\mathcal{G}) \to \mathcal{F}$ is surjective;

(c) for every open affine $U \subseteq Y$, $\mathcal{F}|p^{-1}(U)$ is generated by its sections on $p^{-1}(U)$.

Corollary (3.4.8). — *Under the hypotheses of (3.4.4), suppose that Y is quasi-compact and separated, or its underlying space is Noetherian. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ module of finite type. Then for sufficiently large n, $\mathcal{F}$ is isomorphic to a quotient of an $\mathcal{O}_X$ module of the form $(p^*(\mathcal{G}))(-n)$, where $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$ module of finite type (depending on n).*

3.5. **Functorial behavior.**

(3.5.1). Let $\phi\colon \mathcal{S}' \to \mathcal{S}$ be a homomorphism of graded quasi-coherent $\mathcal{O}_Y$ algebras, and set $X = \operatorname{Proj}(\mathcal{S})$, $X' = \operatorname{Proj}(\mathcal{S}')$, with structure morphisms $p\colon X \to Y$, $p'\colon X' \to Y$. For each open affine $U \subseteq Y$, the homomorphism $\phi_U\colon \Gamma(U, \mathcal{S}') = S'_U \to S_U = \Gamma(U, \mathcal{S})$ induces a U-morphism $\Phi_U\colon G(\phi_U) \to X'_U$, by (2.8.1). For $V \subseteq U$, we have $G(\phi_V) = G(\phi_U) \cap p^{-1}(V)$, and Φ_V is the restriction of Φ_U to $G(\phi_V)$. Hence there is an open set $G(\phi) \subseteq X$ such that $G(\phi) \cap p^{-1}(U) = G(\phi_U)$ for every affine U, and a morphism $\Phi\colon G(\phi) \to X'$ whose restriction to $G(\phi_U)$ is Φ_U.

If every $y \in Y$ has a neighborhood U such that $\phi_U((S'_U)_+)$ generates $(S_U)_+$ [or more generally, such that the radical of the ideal it generates contains $(S_U)_+$], then $G(\phi) = X$.

Proposition (3.5.2). — *(i) [see (2.8.7)] If $\mathcal{M}$ is a quasi-coherent graded $\mathcal{S}$ module, then $(\mathcal{M}_{[\phi]})^{\sim} \cong \Phi_*(\widetilde{\mathcal{M}})$.*

(ii) [see (2.8.8)] If $\mathcal{M}'$ is a quasi-coherent graded $\mathcal{S}'$ module, there is a canonical fucntorial homomorphism $\Phi^*(\widetilde{\mathcal{M}'}) \to (\mathcal{M}' \otimes_{\mathcal{S}'} \mathcal{S})^{\sim}|G(\phi)$. If $\mathcal{S}'_1$ generates $\mathcal{S}'$, it is an isomorphism.

In particular, for each n there is a canonical homomorphism

$$\Phi^*(\mathcal{O}_{X'}(n)) \to \mathcal{O}_X(n)|G(\phi). \tag{3.5.2.1}$$

Proposition (3.5.3). — *Given a morphism $\psi\colon Y' \to Y$, and a quasi-coherent graded $\mathcal{O}_Y$ algebra $\mathcal{S}$, set $\mathcal{S}' = \psi^*\mathcal{S}$. Then* $\operatorname{Proj}(\mathcal{S}') \cong \operatorname{Proj}(\mathcal{S}) \times_Y Y'$, *and if $\mathcal{M}$ is a quasi-coherent graded $\mathcal{S}$ module, then $\psi^*(\mathcal{M})^{\sim} \cong \widetilde{\mathcal{M}} \otimes_Y \mathcal{O}_{Y'}$.*

Corollary (3.5.4). — *In the setting of (3.5.3), $\mathcal{O}_{X'}(n) \cong \mathcal{O}_X(n) \otimes_Y Y'$, where $X' =$* $\operatorname{Proj}(\mathcal{S}')$*, $X =$* $\operatorname{Proj}(\mathcal{S})$*.*

(3.5.5). Keeping the preceding notation, let $\Psi\colon X' \to X$ be the canonical morphism, and set $\mathcal{M}' = \psi^*(\mathcal{M})$. Assume that $\mathcal{S}_1$ generates $\mathcal{S}$ and that X is of finite type over Y; then the same hold for $\mathcal{S}'$, X', Y'. Given an $\mathcal{O}_X$ module $\mathcal{F}$, set $\mathcal{F}' = \Psi^*(\mathcal{F})$. By (3.5.4) and (0, 4.3.3), we have $\mathcal{F}'(n) = \Psi^*(\mathcal{F}(n))$. Let

$$q\colon X \to Y, \qquad q'\colon X' \to Y'$$

be the structure morphisms. The canonical homomorphism $\mathcal{F}(n) \to \Psi_*(\Psi^*(\mathcal{F}(n))) = \Psi_*(\mathcal{F}'(n))$ gives rise to $q_*(\mathcal{F}(n)) \to q_*(\Psi_*(\mathcal{F}'(n))) = \psi_*(q'_*(\mathcal{F}'(n)))$. Hence we have a canonical Ψ-homomorphism $\theta\colon \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{F}')$. Then (2.8.13.1-2) yield commutative diagrams

$$\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \mathcal{F}' \\
{\scriptstyle \beta_{\mathcal{F}}}\big\uparrow & & \big\uparrow{\scriptstyle \beta_{\mathcal{F}'}} \\
\Gamma_*(\mathcal{F})^{\sim} & \xrightarrow{\widetilde{\theta}} & \Gamma_*(\mathcal{F}'),
\end{array}$$

$$\begin{array}{ccc}
\Gamma_*(\widetilde{\mathcal{M}}) & \xrightarrow{\theta} & \Gamma_*(\widetilde{\mathcal{M}'}) \\
{\scriptstyle \alpha_{\mathcal{M}}}\big\uparrow & & \big\uparrow{\scriptstyle \alpha_{\mathcal{M}'}} \\
\mathcal{M} & \longrightarrow & \mathcal{M}'
\end{array},$$

where the unlabelled horizontal arrows are the canonical Ψ- or ψ-morphisms.

(3.5.6). Now suppose given a morphism $g\colon Y' \to Y$, a graded quasi-coherent $\mathcal{O}_Y$ algebra (resp. $\mathcal{O}_{Y'}$ algebra) $\mathcal{S}$ (resp. $\mathcal{S}'$), and a g-homomorphism of graded algebras $u\colon \mathcal{S} \to \mathcal{S}'$ (*i.e.*, a homomorphism $u\colon \mathcal{S} \to g_*(\mathcal{S}')$, or equivalently $u^\sharp\colon g^*(\mathcal{S}) \to \mathcal{S}'$). This gives a Y'-morphism $G(u^\sharp) \to \operatorname{Proj}(g^*(\mathcal{S})) = X \times_Y Y'$, where $X = \operatorname{Proj}(\mathcal{S})$, and $G(u^\sharp)$ is open in $X' = \operatorname{Proj}(\mathcal{S}')$. Composing with the projection of $X \times_Y Y'$ on X, we get a morphism $v\colon G(u^\sharp) \to X$, denoted $v = \operatorname{Proj}(u)$, and commutative diagram

$$\begin{array}{ccc}
G(u^\sharp) & \xrightarrow{v} & X \\
\big\downarrow & & \big\downarrow \\
Y' & \xrightarrow{g} & Y.
\end{array}$$

To any quasi-coherent graded $\mathcal{S}$ module $\mathcal{M}$ there corresponds a canonical υ-morphism

$$\upsilon\colon \widetilde{\mathcal{M}} \to (g^*(\mathcal{M}) \otimes_{g^*(\mathcal{S})} \mathcal{S}')^{\sim}|G(u^\sharp), \tag{3.5.6.1}$$

and if $\mathcal{S}_1$ generates $\mathcal{S}$, then $\upsilon^\sharp$ is an isomorphism. In particular, we have

$$\upsilon\colon \mathcal{O}_X(n) \to \mathcal{O}_{X'}(n)|G(u^\sharp). \tag{3.5.6.2}$$

3.6. **Closed subschemes of** $\mathrm{Proj}(\mathcal{S})$**.**

(3.6.1). Using (3.1.8), the analog of (2.9.1) holds for a homomorphism of graded quasi-coherent $\mathcal{O}_Y$-alebras $\phi\colon \mathcal{S} \to \mathcal{S}'$.

Proposition (3.6.2). — *[see (2.9.2)] Let* $X = \mathrm{Proj}(\mathcal{S})$.

(i) If $\phi\colon \mathcal{S} \to \mathcal{S}'$ is (TN)-surjective, then the associated morphism $\Phi = \mathrm{Proj}(\phi)$ *(3.5.1) is defined on all of* $\mathrm{Proj}(S')$ *and is a closed immersion into X. If $\mathcal{I} = \ker(\phi)$, the image of Φ is the closed subscheme defined by the ideal sheaf $\widetilde{\mathcal{I}} \subseteq \mathcal{O}_X$.*

(ii) Suppose further that $\mathcal{S}_0 = \mathcal{O}_Y$, $\mathcal{S}_1$ generates $\mathcal{S}$, and $\mathcal{S}_1$ is of finite type. Let $X' \subseteq X$ be a closed subscheme, defined by a quasi-coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$, and let $\mathcal{J} \subseteq S$ be the preimage of $\Gamma_*(\mathcal{I})$ under $\alpha\colon \mathcal{S} \to \Gamma_*(\mathcal{O}_X)$ (3.3.2). Set $\mathcal{S}' = \mathcal{S}/\mathcal{J}$. Then X' is the image of the closed immersion $\mathrm{Proj}(\mathcal{S}') \to X$ *associated to the canonical surjection $\mathcal{S} \to \mathcal{S}'$.*

Corollary (3.6.3). — *In (3.6.2, (i)), if $\mathcal{S}_1$ generates $\mathcal{S}$, then $\Phi^*(\mathcal{O}_X(n)) = \mathcal{O}_{X'}(n)$ [see (2.9.3)].*

Corollary (3.6.4). — *Let $\mathcal{S}$ be a quasi-coherent sheaf of graded $\mathcal{O}_Y$ algebras such that $\mathcal{S}_1$ generates $\mathcal{S}$, let $u\colon \mathcal{M} \to \mathcal{S}_1$ be a surjective homomorphism of quasi-coherent $\mathcal{O}_Y$ modules, and let $\overline{u}\colon \mathbf{S}_{\mathcal{O}_Y}(\mathcal{M}) \to \mathcal{S}$ be the graded algebra homomorphism that extends u (1.7.4). Then the morphism* $\mathrm{Proj}(\overline{u})$ *is a closed immersion of* $\mathrm{Proj}(\mathcal{S})$ *into* $\mathrm{Proj}(\mathbf{S}_{\mathcal{O}_Y}(\mathcal{M}))$.

3.7. **Morphisms from a prescheme to a homogeneous spectrum.**

(3.7.1). Let $q\colon X \to Y$ be a morphism of preschemes, $\mathcal{L}$ an invertible $\mathcal{O}_X$ module, $\mathcal{S}$ a graded quasi-coherent $\mathcal{O}_Y$ algebra; then $q^*(\mathcal{S})$ is a graded quasi-coherent $\mathcal{O}_X$ algebra. Suppose given a graded $\mathcal{O}_X$ algebra homomorphism

$$\psi\colon q^*(\mathcal{S}) \to \mathcal{S}' = \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n},$$

or equivalently, a q-morphism of graded algebras

$$\psi^\flat\colon \mathcal{S} \to q_*(\mathcal{S}').$$

Now, $\mathrm{Proj}(\mathcal{S}') = X$, by (3.1.7) and (3.1.8, (iii)), so we get an open subset $G(\psi) \subseteq X$ and a Y-morphism

$$r_{\mathcal{L},\psi}\colon G(\psi) \to \mathrm{Proj}(\mathcal{S}) = P \tag{3.7.1.1}$$

associated to $\mathcal{L}$ and ψ, as in (3.5.6).

(3.7.2). Let us describe $r = r_{\mathcal{L},\psi}$ more explicitly when $Y = \operatorname{Spec}(A)$ is affine, so $\mathcal{S} = \widetilde{S}$. First suppose $X = \operatorname{Spec}(B)$ affine and $\mathcal{L} = \widetilde{L}$, where L is a free B module of rank 1, with generator c, say. Then ψ corresponds to a graded A algebra homomorphism $S \otimes_A B \to B[c]$, necessarily of the form $(s \otimes b) \mapsto bv(s)c^n$ for $s \in S_n$, where $v\colon S \to B$ is an (ungraded) A algebra homomorphism. Given $f \in S_d$, set $g = v(f)$. Then $r^{-1}(D_+(f)) = D(g)$, and the restriction $r\colon D(g) \to D_+(f)$ corresponds to the ring homomorphism $S_{(f)} \subseteq S_f \to B_g$ induced by v. Here $G(\psi)$ is the union of such open sets $D(g) \subseteq X$. The generalization to arbitrary X (Y still affine) is as follows.

Proposition (3.7.3). — *If $Y = \operatorname{Spec}(A)$ is affine and $\mathcal{S} = \widetilde{S}$, then for every $f \in S_d$, we have*

$$r_{\mathcal{L},\psi}^{-1}(D_+(f)) = X_{\psi^\flat(f)} \qquad (\textit{where } \psi^\flat(f) \in \Gamma(X, \mathcal{L}^{\otimes d})) \tag{3.7.3.1}$$

and the restriction $X_{\psi^\flat(f)} \to D_+(f) = \operatorname{Spec}(S_{(f)})$ corresponds (I, 2.2.4) to the algebra homomorphism

$$\psi_f^\flat\colon S_{(f)} \to \Gamma(X_{\psi^\flat(f)}, \mathcal{O}_X) \tag{3.7.3.2}$$

given, for $s \in S_{nd}$, by

$$\psi_{(f)}^\flat(s/f^n) = (\psi^\flat(s)|X_{\psi^\flat(f)})/(\psi^\flat(f)|X_{\psi^\flat(f)})^n. \tag{3.7.3.3}$$

Note that $G(\psi)$ is the union of the open sets $X_{\psi^\flat(f)}$ for $f \in S_d$, $d > 0$. We say that $r_{\mathcal{L},\psi}$ is *defined everywhere* if $G(\psi) = X$. This property is local with respect to Y.

Corollary (3.7.4). — *Under the hypotheses of (3.7.3), $r_{\mathcal{L},\psi}$ is defined everywhere if and only if for every $x \in X$ there exists $d > 0$ and $s \in S_d$ such that $t = \psi^\flat(s) \in \Gamma(X, \mathcal{L}^{\otimes n})$ satisfies $t(x) \neq 0$.*

This condition always holds if ψ is (TN)-surjective.

Similarly, the property that $r_{\mathcal{L},\psi}$ is *dominant* is local on Y, and for Y affine, we have:

Corollary (3.7.5). — *Under the hypotheses of (3.7.3), $r_{\mathcal{L},\psi}$ is dominant if and only if for every $n > 0$, every $s \in S_n$ such that $\psi^\flat(s) \in \Gamma(X, \mathcal{L}^{\otimes n})$ is locally nilpotent, is itself nilpotent.*

Proof: the condition says that if $r_{\mathcal{L},\psi}^{-1}(D_+(s))$ is empty, then $D_+(s)$ is empty [see (2.3.7)].

Proposition (3.7.6). — *Given a morphism $q\colon X \to Y$, an invertible $\mathcal{O}_X$ module $\mathcal{L}$, quasi-coherent graded $\mathcal{O}_Y$ algebras $\mathcal{S}$, $\mathcal{S}'$, and algebra homomorphisms $u\colon \mathcal{S}' \to \mathcal{S}$, $\psi\colon q^*(\mathcal{S}) \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$, let $\psi' = \psi \circ q^*(u)$. If $r_{\mathcal{L},\psi'}$ is defined everywhere, then so is $r_{\mathcal{L},\psi}$. If u is (TN)-surjective and $r_{\mathcal{L},\psi'}$ is dominant, then so is $r_{\mathcal{L},\psi}$. Conversely, if u is (TN)-injective and $r_{\mathcal{L},\psi}$ is dominant, then so is $r_{\mathcal{L},\psi'}$.*

Proposition (3.7.7). — *Let Y be a quasi-compact prescheme, $q\colon X \to Y$ a quasi-compact morphism, $\mathcal{L}$ an invertible $\mathcal{O}_X$ module, $\mathcal{S}$ a quasi-coherent graded $\mathcal{O}_Y$ algebra, $\psi\colon q^*(\mathcal{S}) \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ an algebra homomorphism. Suppose $\mathcal{S}$ is the inductive limit of a filtered system of quasi-coherent graded $\mathcal{O}_Y$ algebras $(\mathcal{S}^\lambda)$, and set $\psi_\lambda = \psi \circ q^*(\phi_\lambda)$, where $\phi_\lambda\colon \mathcal{S}^\lambda \to \mathcal{S}$ is the canonical homomorphism. Then $r_{\mathcal{L},\psi}$ is defined everywhere if and only if some $r_{\mathcal{L},\psi_\lambda}$ is defined everywhere; in that case $r_{\mathcal{L},\psi_\mu}$ is defined everywhere for all $\mu \geq \lambda$.*

Corollary (3.7.8). — *Under the hypotheses of (3.7.7), if the $r_{\mathcal{L},\psi_\lambda}$ are dominant, then so is $r_{\mathcal{L},\psi}$. The converse holds if the ϕ_λ are injective.*

Remarks (3.7.9). — (i) With the notation of (3.7.1), there is a canonical homomorphism

$$\theta\colon r^*_{\mathcal{L},\psi}(\mathcal{O}_P(n)) \to \mathcal{L}^{\otimes n}. \tag{3.7.9.1}$$

defined as in (3.5.6.2).

(ii) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$ module. Suppose q quasi-compact and separated, whence $q_*(\mathcal{F}\otimes\mathcal{L}^{\otimes n})$ is quasi-coherent on Y. Then $\mathcal{M}' = \bigoplus_n \mathcal{F}\otimes\mathcal{L}^{\otimes n}$ is a quasi-coherent graded $\mathcal{S}'$ module, and $\mathcal{M} = q_*(\mathcal{M}') = \bigoplus_n q_*(\mathcal{F}\otimes\mathcal{L}^{\otimes n})$ is a quasi-coherent $\mathcal{S}$ module via $\psi^\flat$. There is a canonical $\mathcal{O}_X$ module homomorphism

$$\xi\colon r^*_{\mathcal{L},\psi}(\widetilde{\mathcal{M}}) \to \mathcal{F}|G(\psi). \tag{3.7.9.2}$$

3.8. **Criteria for immersion into a homogeneous spectrum.**

(3.8.1). With the notation of (3.7.1), the property that $r_{\mathcal{L},\psi}$ is an (open, closed) immersion is local on Y.

Proposition (3.8.2). — *Under the hypotheses of (3.7.3), $r_{\mathcal{L},\psi}$ is defined everywhere and is an immersion if and only if there exist sections $s_\alpha \in S_{n_\alpha}$ ($n_\alpha > 0$) such that, setting $f_\alpha = \psi^\flat(s_\alpha)$, the following hold:*

(i) The open sets X_{f_α} cover X.

(ii) The X_{f_α} are affine.

(iii) For every α and every $t \in \Gamma(X_{f_\alpha}, \mathcal{O}_X)$, there exists $m > 0$ and $s \in S_{mn_\alpha}$ such that $t = (\psi^\flat(s)|X_{f_\alpha})/(f_\alpha|X_{f_\alpha})^m$.

Moreover, $r_{\mathcal{L},\psi}$ is an open immersion if there exists (s_α) satisfying (i)-(iii) and:

(iv) For every $m > 0$ and $s \in mn_\alpha$ such that $\psi^\flat(s)|X_{f_\alpha} = 0$, there exists k such that $s_\alpha^k s = 0$.

Likewise, $r_{\mathcal{L},\psi}$ is a closed immersion if there exists (s_α) satisfying (i)-(iii) and:

(v) The open sets $D_+(s_\alpha)$ cover $P = \mathrm{Proj}(S)$.

Corollary (3.8.3). — *Under the hypotheses of (3.7.6), if $r_{\mathcal{L},\psi'}$ is defined everywhere and is an immersion, then so is $r_{\mathcal{L},\psi}$. If in addition u is (TN)-surjective and $r_{\mathcal{L},\psi'}$ is an open (resp. closed) immersion, then so is $r_{\mathcal{L},\psi}$.*

Proposition (3.8.4). — *Assume the hypotheses of (3.7.7) and also that $q\colon X \to Y$ is of finite type. Then $r_{\mathcal{L},\psi}$ is defined everywhere and is an immersion if and only if the same holds for some $r_{\mathcal{L},\lambda}$, in which case it also holds for $r_{\mathcal{L},\mu}$, for all $\mu \geq \lambda$.*

Proposition (3.8.5). — *Assume that Y is quasi-compact and separated, or that its underlying space is Noetherian. Let $q\colon X \to Y$ be a morphism of finite type, $\mathcal{L}$ an invertible $\mathcal{O}_X$ module, $\mathcal{S}$ a quasi-coherent graded $\mathcal{O}_Y$ algebra, $\psi\colon \mathcal{S} \to \bigoplus_{n\geq 0}\mathcal{L}^{\otimes n}$ a graded algebra homomorphism. Then $r_{\mathcal{L},\psi}$ is defined everywhere and is an immersion if and only if there exist $n > 0$ and a sub-$\mathcal{O}_Y$ module $\mathcal{E} \subseteq \mathcal{S}_n$ of* finite type *such that:*

(a) the homomorphism $\psi_n \circ q^*(j_n)\colon q^*(\mathcal{E}) \to \mathcal{L}^{\otimes n}$ *(where* $j_n\colon \mathcal{E} \to \mathcal{S}_n$ *is the inclusion) is surjective; and*

(b) letting $\mathcal{S}'$ *be the (graded) sub-*$\mathcal{O}_Y$ *algebra of* $\mathcal{S}$ *generated by* $\mathcal{E}$*,* $j'\colon \mathcal{S}' \to \mathcal{S}$ *the inclusion, and* $\psi' = \psi \circ q^*(j')$*,* $r_{\mathcal{L},\psi'}$ *is defined everywhere and is an immersion.*

When these conditions hold, they also hold for every quasi-coherent sub-$\mathcal{O}_Y$ *module* $\mathcal{E} \subseteq \mathcal{E}' \subseteq \mathcal{S}_n$*, and for the image of* $\mathcal{E}^{\otimes k}$ *in* $\mathcal{S}_{kn}$*.*