2.5. Sheaf associated to a graded module.

(2.5.1). If $M$ is a graded $S$ module, then $M_{(f)}$ is an $S_{(f)}$ module, giving a quasi-coherent sheaf $\tilde{M}_{(f)}$ on $\text{Spec}(S_{(f)}) = D_+(f) \subseteq \text{Proj}(S)$ (I, 1.3.4).

Proposition (2.5.2). — Given a graded $S$ module $M$, there is a unique quasi-coherent sheaf $\tilde{M}$ of $\mathcal{O}_X$ modules on $X = \text{Proj}(S)$ such that $\Gamma(D_+(f), \tilde{M}) = M_{(f)}$ for every homogeneous $f \in S_+$, with restriction from $D_+(f)$ to $D_+(fg)$ given by the canonical homomorphism $M_{(f)} \to M_{(fg)}$.

Definition (2.5.3). — $\tilde{M}$ in (2.5.2) is the sheaf associated to the graded $S$ module $M$.

Proposition (2.5.4). — $M \mapsto \tilde{M}$ is an exact functor which commutes with inductive limits and arbitrary direct sums.

Proposition (2.5.5). — For all $p \in \text{Proj}(S)$, we have $\tilde{M}_p = M_{(p)}$.

Proposition (2.5.6). — Suppose that for every $z \in M$ and every homogeneous $f \in S_+$, some power of $f$ annihilates $z$. Then $\tilde{M} = 0$. If $S_1$ generates $S$ as an $S_0$-algebra, the converse holds.

Proposition (2.5.7). — Let $f \in S_d$, $d > 0$. For every integer $n$, the sheaf $S(nd)^\sim |D_+(f)$ is isomorphic to $\mathcal{O}_X|D_+(f)$.

Corollary (2.5.8). — The restriction of $S(nd)^\sim$ to the open set $U = \bigcup_{f \in S_d} D_+(f)$ is invertible [i.e., locally free of rank 1 (0, 5.4.1)].

Corollary (2.5.9). — If $S_1$ generates $S_+$, then $S(n)^\sim$ is an invertible sheaf on $X = \text{Proj}(S)$ for every $n$.

(2.5.10). From now on we use the notation

(2.5.10.1) $\mathcal{O}_X(n) = S(n)^\sim$

and also, for any open $U \subseteq X$ and sheaf of $\mathcal{O}_X|U$ modules $\mathcal{F}$,

(2.5.10.2) $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X|U} (\mathcal{O}(n)|U)$.

If $S_1$ generates $S_+$ then the functor $\mathcal{F} \mapsto \mathcal{F}(n)$ is exact.

(2.5.11). Given graded modules $M$, $N$, there are canonical functorial homomorphisms

(2.5.11.1) $\lambda_{(f)}: M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$

and hence

(2.5.11.2) $\lambda: \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to (M \otimes_S N)^\sim$.

If $\mathcal{I}$ and $\mathcal{J}$ are graded ideals, then since $\tilde{\mathcal{I}}$, $\tilde{\mathcal{J}}$ are ideal sheaves, there is a canonical homomorphism $\tilde{\mathcal{I}} \otimes_{\mathcal{O}_X} \tilde{\mathcal{J}} \to \mathcal{O}_X$. It is equal to the composite

(2.5.11.3) $\tilde{\mathcal{I}} \otimes_{\mathcal{O}_X} \tilde{\mathcal{J}} \to (\mathcal{I} \otimes_{\mathcal{S}} \mathcal{J})^\sim \to \mathcal{O}_X$. 


Finally, given three graded modules, there is a canonical homomorphism

\[ \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \otimes_{\mathcal{O}_X} \tilde{P} \to (M \otimes_{S} N \otimes_{S} P)^\sim \]

given by \( \lambda \circ (\lambda \otimes 1) = \lambda \circ (1 \otimes \lambda) \).

Similarly, there is a canonical functorial homomorphism of \( S(f) \) modules

\[ \mu(f): \text{Hom}_S(M, N)(f) \to \text{Hom}_{S(f)}(M(f), N(f)) \]

and hence, using (I, 1.3.8), a canonical homomorphism of \( \mathcal{O}_X \) module sheaves

\[ \mu: \text{Hom}_S(M, N)^\sim \to \text{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N}). \]

**Proposition (2.5.13).** — Suppose \( S_1 \) generates \( S_+ \). Then \( \lambda \) in (2.5.11.2) is an isomorphism; and if \( M \) is finitely presented (2.1.1), then so is \( \mu \) in (2.5.12.2). If \( I \) is a graded ideal, then \( \tilde{I} \tilde{M} = (\tilde{IM})^\sim \).

**Corollary (2.5.14).** — If \( S_1 \) generates \( S_+ \), then there are canonical isomorphisms

\[ O_X(m) \otimes O_X(n) \cong O_X(m + n) \]
\[ O_X(n) \cong (O_X(1))^\otimes_n \]

for all integers \( m, n \).

**Corollary (2.5.15).** — If \( S_1 \) generates \( S_+ \), then there is a canonical isomorphism \( M(n)^\sim \cong \tilde{M}(n) \), for every graded module \( M \).

Under the identifications \( X = \text{Proj}(S) \cong X' = \text{Proj}(S') \cong X^{(d)} = \text{Proj}(S^{(d)}) \) of (2.4.7), we have \( O_X(n) \cong O_{X'}(n) \) and \( O_{X^{(d)}}(n) \cong O_{X}(nd) \).

**Proposition (2.5.17).** — The canonical homomorphisms \( O_X(n)d \otimes O_X(md) \to O_X((m + nd)) \) restrict to isomorphisms on \( U = \bigcup_{f \in S_d} D_+(f) \).

### 2.6. Graded \( S \) module associated to a sheaf on \( \text{Proj}(S) \)

In this section we assume that \( S_1 \) generates the ideal \( S_+ \), and put \( X = \text{Proj}(S) \).

(2.6.1). By (2.5.9), the sheaf \( O_X(1) \) is invertible. For any \( O_X \) module sheaf \( \mathcal{F} \) we define as in (0, 5.4.6)

\[ \Gamma_*(\mathcal{F}) = \Gamma_*(O_X(1), \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)), \]

the second equality following from (2.5.14.2). Then (0, 5.4.6) \( \Gamma_*(O_X) \) is a graded ring and \( \Gamma_*(\mathcal{F}) \) is a graded \( \Gamma_*(O_X) \) module sheaf. Since \( O_X(n) \) is locally free, \( \mathcal{F} \mapsto \Gamma_*(\mathcal{F}) \) is a left exact functor. In particular, if \( \mathcal{I} \) is an ideal sheaf, then \( \Gamma_*(\mathcal{I}) \) is a graded ideal in \( \Gamma_*(O_X) \).
(2.6.2). The map \( x \mapsto x/1: M_0 \to M(f) \) induces maps \( M_0 \to \Gamma(D_+(f), \tilde{M}) \) for all homogeneous \( f \in S_+ \), compatible with restrictions, and hence a map

\( \alpha_0: M_0 \to \Gamma(X, \tilde{M}) \).

Applying this to \( M(n) \) and using (2.5.15), we get

\( \alpha_n: M_n = M(n)_0 \to \Gamma(X, \tilde{M}(n)) \),

and hence a homomorphism of graded abelian groups

\( \alpha: M \to \Gamma_*(\tilde{M}) \).

The map \( \alpha: S \to \Gamma_*(\mathcal{O}_X) \) is a graded ring homomorphism, and (2.6.2.3) is an \( S \) module homomorphism.

Proposition (2.6.3). — For every \( f \in S_d \ (d > 0) \), the open set \( D_+(f) \) is the non-vanishing locus of the section \( \alpha(f) \) of \( \mathcal{O}_X(d) \) (0, 5.5.2).

(2.6.4). Set \( M = \Gamma_*(\mathcal{F}) \), which we may consider as an \( S \) module via \( S \to \Gamma_*(\mathcal{O}_X) \). By (2.6.3), the section \( \alpha_d(f) \) of \( \mathcal{O}_X(d) \) is invertible on \( D_+(f) \). Hence there is an \( S(f) \) module homomorphism

\( \beta(f): M(f) \to \Gamma(D_+(f), \mathcal{F}) \)

given by \( z/f^n \mapsto (z|D_+(f))/(\alpha_d(f)|D_+(f))^n \). This is compatible with restriction to \( D_+(fg) \), giving a canonical homomorphism of sheaves of \( \mathcal{O}_X \) modules

\( \beta: \Gamma_*(\mathcal{F}) \to \mathcal{F} \).

Proposition (2.6.5). — For any graded \( S \) module \( M \) and sheaf of \( \mathcal{O}_X \) modules \( \mathcal{F} \), each of the following maps is the identity:

\[
\begin{align*}
\tilde{M} & \xrightarrow{\alpha} \Gamma_*(\tilde{M}) \xrightarrow{\beta} \tilde{M}, \\
\Gamma_*(\mathcal{F}) & \xrightarrow{\alpha} \Gamma_*(\Gamma_*(\mathcal{F})^*) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F})
\end{align*}
\]

2.7. Finiteness conditions.

Proposition (2.7.1). — (i) If \( S \) is a Noetherian graded ring, then \( X = \text{Proj}(S) \) is a Noetherian scheme.

(ii) If \( S \) is a finitely-generated graded \( A \)-algebra, then \( X \) is a scheme of finite type over \( Y = \text{Spec}(A) \).

(2.7.2). Consider two conditions on a graded \( S \) module \( M \):

(TF) There exists \( n \) such that \( \bigoplus_{k \geq n} M_k \) is a finitely generated \( S \) module;

(TN) There exists \( n \) such that \( M_k = 0 \) for \( k \geq n \).

A graded \( S \) module homomorphism \( u \) will be called (TN)-injective (resp. (TN)-surjective, (TN)-bijective) if its kernel (resp. cokernel, both) satisfies (TN). By (2.5.4), this implies that \( \tilde{u} \) is injective (resp. surjective, bijective).
Proposition (2.7.3). — Assume that $S_+$ is a finitely generated ideal. 
(i) If $M$ satisfies (TF), then $\tilde{M}$ is an $O_X$ module of finite type. 
(ii) If $M$ satisfies (TF), then $\tilde{M} = 0$ if and only if $M$ satisfies (TN).

Corollary (2.7.4). — If $S_+$ is finitely generated, then $\text{Proj}(S) = \emptyset$ iff there is an $n$ such that $S_k = 0$ for all $k \geq n$.

Theorem (2.7.5). — Let $X = \text{Proj}(S)$, where $S_+$ is generated by finitely many elements, homogeneous of degree 1. Then for every quasi-coherent sheaf of $O_X$ modules $F$, the canonical homomorphism $\beta: \Gamma_*(F) \to F$ (2.6.4) is an isomorphism.

Remark (2.7.6). — If $S$ is Noetherian and $S_1$ generates $S_+$, then the hypotheses of (2.7.5) hold.

Corollary (2.7.7). — Under the hypotheses of (2.7.5), every quasi-coherent $O_X$ module $F$ is isomorphic to $\tilde{M}$ for some graded $S$ module $M$.

Corollary (2.7.8). — Under the hypotheses of (2.7.5), every quasi-coherent $O_X$ module $F$ of finite type is isomorphic to $\tilde{N}$ for some finitely generated graded $S$ module $N$.

Corollary (2.7.9). — Under the hypotheses of (2.7.5), let $F$ be a quasi-coherent $O_X$ module of finite type. Then there exists $n_0$ such that for all $n \geq n_0$, $F(n)$ is isomorphic to a quotient of $O_X^k$ (where $k$ depends on $n$), i.e., $F(n)$ is generated by finitely many global sections $(0, 5.1.1)$.

Corollary (2.7.10). — Under the hypotheses of (2.7.5), let $F$ be a quasi-coherent $O_X$ module of finite type. Then there exists $n_0$ such that for all $n \geq n_0$, $F$ is isomorphic to a quotient of $O_X(-(n))^k$ (where $k$ depends on $n$).

Proposition (2.7.11). — Assume the hypotheses of (2.7.5) hold, and let $M$ be a graded $S$ module.

(i) The canonical homomorphism $\tilde{\alpha}: \tilde{M} \to \Gamma_*(\tilde{M})$ is an isomorphism.

(ii) Let $G \subseteq \tilde{M}$ be a quasi-coherent $O_X$ submodule sheaf, and let $N \subseteq M$ be the preimage of $\Gamma_*(G) \subseteq \Gamma_*(\tilde{M})$ via $\alpha$. Then $\tilde{N} = G$.

2.8. Functorial behavior.

(2.8.1). Let $\phi: S' \to S$ be a graded ring homomorphism. Let $G(\phi)$ denote the complement of $V_+(\phi(S'_+))$ in $X = \text{Proj}(S)$, that is, the union of the open sets $D_+(\phi(f'))$ for homogeneous $f' \in S_+$. Then $a\phi: \text{Spec}(S) \to \text{Spec}(S')$ induces a continuous map $a\phi: G(\phi) \to \text{Proj}(S')$ such that

(2.8.1.1) $a\phi^{-1}(D_+(f')) = D_+(\phi(f'))$.

Let $f = \phi(f')$. Then $\phi$ induces $\phi_1: S'_f \to S_{\phi(f')}$ and $\phi(\phi): S'_f \to S_{(\phi(f'))}$, hence a morphism $a\phi_1: D_+(f) \to D_+(f')$, which on the underlying space is the restriction of $a\phi$ to the open sets in (2.8.1.1). These are compatible with restriction to $D_+(fg)$. 
Proposition (2.8.2). — There is a unique morphism \((\phi', \tilde{\phi})\): \(G(\phi) \to \text{Proj}(S')\) (called the morphism associated to \(\phi\) and denoted \(\text{Proj}(\phi)\)) whose restriction to each \(D_+(\phi(f'))\) coincides with \(\phi'\).

Corollary (2.8.3). — (i) \(\text{Proj}(\phi)\) is an affine morphism.

(ii) If \(\ker(\phi)\) is nilpotent (in particular, if \(\phi\) is injective), then \(\text{Proj}(\phi)\) is dominant.

In general a morphism \(\text{Proj}(S) \to \text{Proj}(S')\) need not be affine, hence not of the form \(\text{Proj}(\phi)\). An example is \(\text{Proj}(S) \to \text{Spec}(A) = \text{Proj}(A[t])\) when \(S\) is an \(A\)-algebra.

(2.8.4). Given a third ring \(S''\) and \(\phi'': S'' \to S'\), let \(\phi'' = \phi \circ \phi'\). Then \(G(\phi'') \subseteq G(\phi)\), and if \(\Phi, \Phi', \Phi''\) are the associated morphisms, then \(\Phi'' = \Phi' \circ (\Phi|G(\phi''))\).

(2.8.5). Suppose \(S\) (resp. \(S'\)) is a graded \(A\)-algebra (resp. \(A'\)-algebra), and \(\psi: A' \to A\) commutes with \(\phi: S' \to S\). Then \(G(\phi)\) and \(\text{Proj}(S')\) are schemes over \(\text{Spec}(A)\) and \(\text{Spec}(A')\) respectively, and the the corresponding diagram commutes.

(2.8.6). Let \(M\) be a graded \(S\) module, which we may consider as a graded \(S'\) module \(M_{[\phi]}\).

Proposition (2.8.7). — There is a canonical functorial isomorphism \((M_{[\phi]}) \sim \Phi_*(\tilde{M}|G(\phi))\), where \(\Phi = \text{Proj}(\phi)\).

Proposition (2.8.8). — Let \(M'\) be a graded \(S'\) module. There is a canonical functorial homomorphism \(\nu: \Phi^*(\tilde{M'}) \to (M' \otimes_{S'} S)|G(\phi)\). If \(S'_1\) generates \(S'_+\), then \(\nu\) is an isomorphism.

(2.8.9). Let \(\psi: A' \to A\) be a ring homomorphism, \(\Psi: Y = \text{Spec}(A) \to \text{Spec}(A') = Y'\) its associated morphism. Let \(S'\) be a positively graded \(A'\)-algebra; then \(S = S' \otimes_{A'} A\) is a positively graded \(A\)-algebra. We have the ring homomorphism \(\phi: S' \to S, \phi(s') = s' \otimes 1\), and \(\phi(S'_+)\) generates \(S_+\) as an \(A\) module, hence \(G(\phi) = \text{Proj}(S) = X\). Set \(X' = \text{Proj}(S')\).

Further, let \(M'\) be a graded \(S'\) module, and set \(M = M' \otimes_{A'} A = M' \otimes_{S'} S\).

Proposition (2.8.10). — With the notation of (2.8.9), we have \(X = X' \times_{Y'} Y,\) and the canonical homomorphism \(\nu: \Phi^*(\tilde{M'}) \to \tilde{M} (2.8.8)\) is an isomorphism.

Corollary (2.8.11). — For all \(n \in \mathbb{Z}\), \(\tilde{M}(n)\) is identified with \(\Phi^*(\tilde{M}'(n)) = \tilde{M}'(n) \otimes_{Y'} \mathcal{O}_Y\). In particular, \(\mathcal{O}_X(n) = \Phi^* \mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(n) \otimes_{Y'} \mathcal{O}_Y\).

(2.8.12). For \(f' \in S'_d (d > 0)\) and \(f = \phi(f')\), the canonical map \(M'_{(f')} \to M_{(f)}\) is identified with \(M''/(f' - 1)M''(d) \to M''(d)/(f - 1)M''(d)\) by (2.2.5).

(2.8.13). In the setting of (2.8.9), let \(\mathcal{F}'\) be an \(\mathcal{O}_{X'}\) module, and set \(\mathcal{F} = \Phi^*(\mathcal{F}')\). Then \(\mathcal{F}(n) = \Phi^*(\mathcal{F}'(n))\) by (2.8.11) and (0. 4.3.3). From (0. 4.4.3) we have \(\Gamma(\rho): \Gamma(X', \mathcal{F}'(n)) \to \Gamma(X, \mathcal{F}(n))\) for all \(n \in \mathbb{Z}\), giving a homomorphism of graded modules \(\Gamma_*(\mathcal{F}') \to \Gamma_*(\mathcal{F})\).
If $S_1$ generates $S_+$ and $\mathcal{F}' = \widetilde{M}'$, then $\mathcal{F} = \widetilde{M}$, where $M = M' \otimes_{A'} A$, and we have commutative diagrams

\[
\begin{align*}
M' & \xrightarrow{\alpha_{M'}} \Gamma_*(\widetilde{M}') \\
\downarrow & \downarrow \\
M & \xrightarrow{\alpha_M} \Gamma_*(\widetilde{M}),
\end{align*}
\]
(2.8.13.1)

\[
\begin{align*}
\Gamma_*(\mathcal{F}') & \xrightarrow{\beta_{\mathcal{F}'}^*} \mathcal{F}' \\
\downarrow & \downarrow \\
\Gamma_*(\mathcal{F}) & \xrightarrow{\beta_{\mathcal{F}}} \mathcal{F},
\end{align*}
\]
(2.8.13.2)

in which the vertical arrows are $\Phi$-morphisms.

(2.8.14). Given a second graded $S'$ module $N'$, we have a canonical homomorphism

\[
(2.8.14.1) \quad \Phi^*((M' \otimes_{S'} N')^\sim) \to (M \otimes_S N)^\sim,
\]

and a commutative diagram

\[
\begin{align*}
\Phi^*(\widetilde{M}' \otimes_{\mathcal{O}_X'} \widetilde{N}') & \xrightarrow{\sim} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \\
\Phi^*(\lambda) & \downarrow \lambda \\
\Phi^*((M' \otimes_{S'} N')^\sim) & \to (M \otimes_S N)^\sim,
\end{align*}
\]
(2.8.14.2)

where the top row is the canonical isomorphism (0, 4.3.3). If $S'_1$ generates $S'_+$, then $S_1$ generates $S_+$, the vertical arrows are isomorphisms by (2.5.13), and hence (2.8.14.1) is an isomorphism.

Similarly, there is a commutative diagram

\[
\begin{align*}
\Phi^*(\text{Hom}_{S'}(M', N')^\sim) & \to \text{Hom}_{S}(M, N)^\sim \\
\Phi^*(\mu) & \downarrow \mu \\
\Phi^*(\text{Hom}_{\mathcal{O}_X'}(\widetilde{M}', \widetilde{N}')) & \to \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}),
\end{align*}
\]

with bottom row given by (0, 4.4.6) and vertical arrows by (2.5.12).

(2.8.15). One can replace $S_0$ and $S'_0$ by $\mathbb{Z}$, or replace $S$ and $S'$ by $S^{(d)}$ and $S'^{(d)}$ as in (2.4.7), without changing $\Phi$.

2.9. **Closed subschemes of** $\text{Proj}(S)$.

(2.9.1). If $\phi: S' \to S$ is (TN)-injective (resp. (TN)-surjective, (TN)-bijective) (2.7.2), then (2.8.15) shows that where $\Phi$ is concerned we can reduce to the case that $\phi$ is actually injective (resp. surjective, bijective).
Proposition (2.9.2). — Let $X = \text{Proj}(S)$.

(i) If $\phi: S \to S'$ is (TN)-surjective, then the associated morphism $\Phi$ is defined on all of $\text{Proj}(S')$ and is a closed immersion into $X$. If $\mathcal{I} = \ker(\phi)$, the image of $\Phi$ is the closed subscheme defined by the ideal sheaf $\tilde{\mathcal{I}}$.

(ii) Suppose further that $S_+$ is generated by finitely many elements, homogeneous of degree 1. Let $X' \subseteq X$ be a closed subscheme, defined by a quasi-coherent sheaf of ideals $\mathcal{J}$, and let $\mathcal{I} \subseteq S$ be the preimage of $\Gamma_*(\mathcal{J})$ under $\alpha: S \to \Gamma_*(\mathcal{O}_X)$ (2.6.2). Set $S' = S/\mathcal{I}$. Then $X'$ is the image of the closed immersion $\text{Proj}(S') \to X$ associated to the canonical surjection $S \to S'$.

Corollary (2.9.3). — In (2.9.2 (i)), if $S_1$ generates $S_+$, then $\Phi^*(S(n)^-) = S'(n)^-$ for all $n$, and $\Phi^*(\mathcal{F}(n)) = (\Phi^*(\mathcal{F}))(n)$ for every $\mathcal{O}_X$ module sheaf $\mathcal{F}$.

Corollary (2.9.4). — In (2.9.2 (ii)), the subscheme $X'$ is integral if and only if the ideal $\mathcal{I}$ is prime.

[“If” is clear from (2.4.4). “Only if” uses (I, 7.4.4).]

Corollary (2.9.5). — Let $S$ be a graded $A$-algebra which is generated by $S_1$, $M$ an $A$ module, and $u: M \to S_1$ a surjective $A$ module homomorphism, inducing $\overline{u}: S(M) \to S$, where $S(M)$ is the symmetric algebra of $M$. Then $\overline{u}$ induces a closed immersion of $\text{Proj}(S)$ into $\text{Proj}(S(M))$. 