Synopsis of material from EGA Chapter II, §1

1. Affine morphisms

1.1. S-preschemes and $\mathcal{O}_S$-algebras.

(1.1.1). Given an S-prescheme $f\colon X \to S$, $\mathcal{A}(X)$ denotes the sheaf of $\mathcal{O}_S$ algebras $f_*\mathcal{O}_X$. Given a sheaf of $\mathcal{O}_X$ modules (or $\mathcal{O}_X$ algebras) $\mathcal{F}$, $\mathcal{A}(\mathcal{F})$ denotes the sheaf of $\mathcal{A}(X)$-modules (or $\mathcal{A}(X)$ algebras) $f_*(\mathcal{F})$.

(1.1.2–3). $X \mapsto \mathcal{A}(X)$ is a contravariant functor from S-preschemes to sheaves of $\mathcal{O}_S$ algebras. More generally, there is a contravariant functor $(X, \mathcal{F}) \mapsto (\mathcal{A}(X), \mathcal{A}(\mathcal{F}))$ from pairs consisting of an S-prescheme X and sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ to pairs consisting of a sheaf of $\mathcal{O}_S$ algebras and a sheaf of modules over it.

1.2. Preschemes affine over a prescheme.

Definition (1.2.1). — An S-prescheme $f\colon X \to S$ is *affine over* S if S has an affine open covering (S_α) such that each $f^{-1}(S_\alpha)$ is affine.

Example (1.2.2). — By (I, 4.2.3-4) any closed sub-prescheme of S is affine over S.

Remark (1.2.3). — A prescheme affine over S need not be affine, e.g., $X = S$. An affine scheme X that is a prescheme over S need not be affine over S (see (1.3.3)), but if S is a *scheme* [*i.e.*, a separated presecheme] then any S-prescheme which is an affine scheme is affine over S (I, 5.5.10).

Proposition (1.2.4). — *Every prescheme affine over S is separated over S, i.e., it is a scheme over S.*

Proposition (1.2.5). — *If $f\colon X \to S$ is affine, then for every open $U \subseteq S$, $f^{-1}(U)$ is affine over U.*

Proposition (1.2.6). — *If $f\colon X \to S$ is affine, then for every quasi-coherent sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$, $f_*(\mathcal{F})$ is quasi coherent.*

In particular, $\mathcal{A}(X)$ is a quasi-coherent sheaf of $\mathcal{O}_S$ algebras.

Proposition (1.2.7). — *Let X be affine over S. For every S-prescheme Y, the canonical map* $\mathrm{Hom}_S(Y, X) \to \mathrm{Hom}_{\mathcal{O}_S\text{-}Alg}(\mathcal{A}(X), \mathcal{A}(Y))$ *is bijective.*

Corollary (1.2.8). — *If X and Y are affine over S, then an S-morphism $h\colon X \to Y$ is an isomorphism iff it induces an isomorphism $\mathcal{A}(X) \cong \mathcal{A}(Y)$.*

1.3. Prescheme affine over S associated to an $\mathcal{O}_S$ algebra.

Proposition (1.3.1). — *Given any quasi-coherent sheaf of $\mathcal{O}_S$ algebra $\mathcal{B}$, there exists a prescheme X affine over S, unique up to canonical isomorphism, such that $\mathcal{A}(X) = \mathcal{B}$.*

The prescheme X in the proposition is denoted $\mathrm{Spec}(\mathcal{B})$.

Corollary (1.3.2). — *Let $f\colon X \to S$ be affine. For every affine $U \subseteq S$, $f^{-1}(U)$ is an affine scheme* $\mathrm{Spec}(\Gamma(U, \mathcal{A}(X)))$.

Example (1.3.3). — Let K be a field, S the affine plane with the origin doubled, so $S = Y_1 \cup Y_2$, where each $Y_i \cong \mathbb{A}^2_K$. Let f be the open immersion $Y_1 \hookrightarrow S$. Then $f^{-1}(Y_2)$ is not affine, so Y_1 is not affine over S, even though Y_1 is an affine scheme.

Corollary (1.3.4). — *Let S be an affine scheme. Then an S-prescheme X is affine over S iff X is an affine scheme.*

Corollary (1.3.5). — *Let X be affine over S and let Y be an X-prescheme. Then Y is affine over X iff Y is affine over S.*

(1.3.6). Let X be affine over S. To give an S-prescheme Y affine over X, it is equivalent to give a quasi-coherent sheaf of $\mathcal{O}_S$ algebras $\mathcal{B}$ and a homomorphism $\mathcal{A}(X) \to \mathcal{B}$; that is, to give a quasi-coherent sheaf of $\mathcal{A}(X)$ algebra on S.

Corollary (1.3.7). — *Let X be affine over S. Then X is of finite type over S iff $\mathcal{A}(X)$ is of finite type as a sheaf of $\mathcal{O}_S$ algebras (I, 9.6.2).*

Corollary (1.3.8). — *A prescheme X affine over S is reduced iff $\mathcal{A}(X)$ is reduced (0, 4.1.4).*

1.4. **Quasi-coherent sheaves on a prescheme affine over S.**

Proposition (1.4.1). — *Let X be affine over S, Y any S-prescheme, $\mathcal{F}$, $\mathcal{G}$ quasi-coherent sheaves of $\mathcal{O}_X$, $\mathcal{O}_Y$ modules. The functorial correspondence from morphisms $(h, u)\colon (Y, \mathcal{G}) \to (X, \mathcal{F})$ to di-homomorphisms $(\mathcal{A}(h), \mathcal{A}(u))\colon (\mathcal{A}(X), \mathcal{A}(\mathcal{F})) \to (\mathcal{A}(Y), \mathcal{A}(\mathcal{G}))$ is bijective.*

Corollary (1.4.2). — *In (1.4.1), suppose Y is also affine over S. Then (h, u) is an isomorphism iff $(\mathcal{A}(h), \mathcal{A}(u))$ is an isomorphism.*

Proposition (1.4.3). — *Given quasi-coherent sheaves of $\mathcal{O}_X$ algebras $\mathcal{B}$ and $\mathcal{B}$ modules $\mathcal{M}$, there exists a prescheme X affine over S and a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules, unique up to canonical isomorphism, such that $(\mathcal{A}(X), \mathcal{A}(\mathcal{F})) \cong (\mathcal{B}, \mathcal{M})$.*

The sheaf $\mathcal{F}$ in the proposition is denoted $\widetilde{\mathcal{M}}$.

Corollary (1.4.4). — *$\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is a covariant exact functor, which commutes with direct limits and direct sums.*

Corollary (1.4.5). — *Under the hypotheses of (1.4.3), $\widetilde{\mathcal{M}}$ is an $\mathcal{O}_X$ module of finite type iff $\mathcal{M}$ is a $\mathcal{B}$ module of finite type.*

Proposition (1.4.6). — *Let Y be affine over S and X, X' affine over Y (hence over S (1.3.5)). Then $X \times_Y X' = \operatorname{Spec}(\mathcal{A}(X) \otimes_{\mathcal{A}(Y)} \mathcal{A}(X'))$ is affine over Y (and over S).*

Corollary (1.4.7). — *If $\mathcal{F}$, $\mathcal{F}'$ are quasi-coherent sheaves of $\mathcal{O}_X$, $\mathcal{O}_{X'}$ modules, then $\mathcal{A}(\mathcal{F} \otimes_Y \mathcal{F}') \cong \mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(Y)} \mathcal{A}(\mathcal{F}')$.*

(1.4.8). In particular, taking $X = X' = Y$ affine over S, if $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent sheaves of $\mathcal{O}_X$ modules, then

$$\mathcal{A}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = \mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(X)} \mathcal{A}(\mathcal{G}). \tag{1.4.8.1}$$

If $\mathcal{F}$ is finitely presented, then (I, 1.6.3 and 1.3.12) imply

$$\mathcal{A}(\mathcal{H}om(\mathcal{F},\mathcal{G})) = \mathcal{H}om_{\mathcal{A}(X)}(\mathcal{A}(\mathcal{F}),\mathcal{A}(\mathcal{G})), \tag{1.4.8.2}$$

up to canonical isomorphism.

Remark (1.4.9). — If X, X' are affine over S, then so is $X \coprod X'$.

Proposition (1.4.10). — *Let $\mathcal{B}$ be a quasi-coherent sheaf of $\mathcal{O}_S$ algebras, $X = \operatorname{Spec}(\mathcal{B})$. If $\mathcal{I} \subseteq \mathcal{B}$ is a quasi-coherent sheaf of ideals, then $\widetilde{\mathcal{I}}$ is a quasi-coherent sheaf of ideals in $\mathcal{O}_X$, and the closed subscheme $Y \subseteq X$ which it defines is canonically isomorphic to* $\operatorname{Spec}(\mathcal{B}/\mathcal{I})$.

Put another way, if $h\colon \mathcal{B} \to \mathcal{B}'$ is a surjective homomorphism of quasi-coherent sheaves of $\mathcal{O}_S$ algebras, then the induced morphism $\operatorname{Spec}(\mathcal{B}') \to \operatorname{Spec}(\mathcal{B})$ is a closed immersion.

Proposition (1.4.11). — *Let $\mathcal{B}$ be a quasi-coherent sheaf of $\mathcal{O}_S$ algebras, $X = \operatorname{Spec}(\mathcal{B})$, $f\colon X \to S$ the structure morphism. If $\mathcal{J} \subseteq \mathcal{O}_S$ is a quasi-coherent sheaf of ideals, then $f^*(\mathcal{J})\mathcal{O}_X \cong (\mathcal{J}\mathcal{B})\tilde{}$, canonically.*

1.5. **Change of base prescheme.**

Proposition (1.5.1). — *If X is affine over S, then any base change $X_{(S')}$ is affine over S'.*

Corollary (1.5.2). — *Let $f\colon X \to S$ be affine, $g\colon S' \to S$ any S-prescheme, $X' = X_{(S')}$, $f'\colon X' \to S'$, $g'\colon X' \to X$ the projections (note $g \circ f' = f \circ g'$). For every quasi-coherent $\mathcal{O}_X$-module, there is a canonical isomorphism*

$$u\colon g^*(f_*(\mathcal{F})) \cong f'_*(g'^*(\mathcal{F})). \tag{1.5.2.1}$$

In particular, $\mathcal{A}(X') \cong g^*(\mathcal{A}(X))$.

Remark (1.5.3). — Although (1.5.2) fails if X is not affine over S, a weaker version is valid for coherent sheaves on X when f is proper and S is Noetherian (III, 4.2.4).

Corollary (1.5.4). — *For $f\colon X \to S$ affine and $s \in S$, the fiber $f^{-1}(s)$ is an affine scheme.*

Corollary (1.5.5). — *If X is an S-prescheme via $f\colon X \to S$, and S' is affine over S, then $X' = X_{(S')}$ is affine over X. Moreover $\mathcal{A}(X') \cong f^*(\mathcal{A}(S'))$ and for every quasi-coherent $\mathcal{A}(S')$-module $\mathcal{M}$, $f^*(\mathcal{M}) \cong \mathcal{A}(f'^*(\widetilde{\mathcal{M}}))$, where $f' = f_{(S')}$.*

(1.5.6). Let $q\colon S' \to S$ be a morphism, $\mathcal{B}$, $\mathcal{B}'$ quasi-coherent sheaves of $\mathcal{O}_S$, $\mathcal{O}_{S'}$ algebras, $u\colon \mathcal{B} \to \mathcal{B}'$ a q-morphism (*i.e.* an $\mathcal{O}_S$ algebra homomorphism $\mathcal{B} \to q_*(\mathcal{B}')$). Then u induces a morphism

$$v = \operatorname{Spec}(u)\colon X' = \operatorname{Spec}(\mathcal{B}') \to \operatorname{Spec}(\mathcal{B}) = X,$$

such that the following diagram commutes

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{q} & S \end{array} . \tag{1.5.6.1}$$

(1.5.7). Moreover, if $\mathcal{M}$ is a quasi-coherent $\mathcal{B}$-module, then

$$v^*(\widetilde{\mathcal{M}}) \cong (q^*(\mathcal{M}) \otimes_{q^*(\mathcal{B})} \mathcal{B}')\tilde{}. \tag{1.5.7.1}$$

1.6. **Affine morphisms.**

(1.6.1). A morphism $f\colon X \to Y$ is *affine* if it makes X affine over Y.

Proposition (1.6.2). — *(i) A closed immersion is affine.*
(ii) The composite of affine morphisms is affine.
(iii) If f is affine, so is any base change $f_{(S')}$.
(iv) If f, g are affine, so is $f \times_S g$.
(v) If $g \circ f$ is affine and g is separated, then f is affine.
(vi) If f is affine, then f_{red} is affine.

Corollary (1.6.3). — *If X is an affine scheme and Y is a [separated] scheme, then any morphism $X \to Y$ is affine.*

Proposition (1.6.4). — *Let Y be locally Noetherian and $f\colon X \to Y$ a morphism of finite type. Then f is affine iff f_{red} is affine.*

1.7. **Vector bundle associated to a sheaf of modules.**

(1.7.1). The *symmetric algebra* $\mathbf{S}(E)$ of an A-module E is the quotient of the tensor algebra $\mathbf{T}(E)$ by the relations $x\otimes y - y\otimes x$ for $x, y \in E$. It has the universal property that any A-linear map $E \to B$, where B is a commutative A-algebra, factors uniquely as $E \to \mathbf{S}(E) \to B$. $\mathbf{S}(-)$ is a functor from A-modules to commutative A-algebras; it commutes with direct limits and has $\mathbf{S}(E \oplus F) = \mathbf{S}(E) \otimes_A \mathbf{S}(F)$. $\mathbf{S}(E)$ is graded, with $\mathbf{S}_n(E)$ [the *n-th symmetric power of E*] the A-linear span of products of n elements of E. We have $\mathbf{S}(A^m) \cong A[t_1, \ldots, t_m]$.

(1.7.2). Let $\phi\colon A \to B$ be a ring homomorphism, F a B-module. $F_{[\phi]}$ denotes F regarded as an A-module. The inclusion $F_{[\phi]} \to \mathbf{S}(F)_{[\phi]}$ and the universal property induce a canonical A-algebra homomorphism $\mathbf{S}(F_{[\phi]}) \to \mathbf{S}(F)_{[\phi]}$. Any A-module homomorphism $E \to F_{[\phi]}$ induces $\mathbf{S}(E) \to \mathbf{S}(F)_{[\phi]}$. We also have $\mathbf{S}(E \otimes_A B) = \mathbf{S}(E) \otimes_A B$.

(1.7.3). Let $R \subseteq A$ be a multiplicative set, and $B = R^{-1}A$. Then $\mathbf{S}(R^{-1}E) = R^{-1}\mathbf{S}(E)$, and if $R \subseteq R'$, then $R^{-1}E \to R'^{-1}E$ commutes with $\mathbf{S}(R^{-1}E) \to \mathbf{S}(R'^{-1}E)$.

(1.7.4). Given a ringed space $(S, \mathcal{A})$ and an $\mathcal{A}$-module $\mathcal{E}$, we have a presheaf of $\mathcal{A}$-algebras $U \mapsto \mathbf{S}(\mathcal{E}(U))$. Its associated sheaf is the *symmetric algebra of* $\mathcal{E}$, denoted $\mathbf{S}(\mathcal{E})$ or $\mathbf{S}_{\mathcal{A}}(\mathcal{E})$. It is functorial and has the corresponding universal property as for the symmetric algebra of a module.

We have $\mathbf{S}(\mathcal{E})_s = \mathbf{S}(\mathcal{E}_s)$ (because $\mathbf{S}$ commutes with direct limits) and $\mathbf{S}(\mathcal{E} \oplus \mathcal{F}) = \mathbf{S}(\mathcal{E}) \otimes_{\mathcal{A}} \mathbf{S}(\mathcal{F})$. $\mathbf{S}(\mathcal{E})$ is graded, and $\mathbf{S}(\mathcal{A}) = \mathcal{A}[t] = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ (regarding $\mathbb{Z}$, $\mathbb{Z}[t]$ as constant sheaves on S).

(1.7.5). Given a morphism of ringed spaces $f\colon (S, \mathcal{A}) \to (T, \mathcal{B})$ and a $\mathcal{B}$-module $\mathcal{F}$, we have $\mathbf{S}(f^*\mathcal{F}) \cong f^*\mathbf{S}(\mathcal{F})$, canonically.

Proposition (1.7.6). — *Let $S = \mathrm{Spec}(A)$, $\mathcal{E} = \widetilde{M}$. Then $\mathbf{S}(\mathcal{E}) = \mathbf{S}(M)\tilde{}$.*

Corollary (1.7.7). — *If $\mathcal{E}$ is a quasi-coherent sheaf of $\mathcal{O}_S$ modules on a prescheme S, then $\mathbf{S}(\mathcal{E})$ is a quasi-coherent sheaf of $\mathcal{O}_S$ algebras. If $\mathcal{E}$ is of finite type, then each $\mathbf{S}_n(\mathcal{E})$ is of finite type.*

Definition (1.7.8). — $\mathbf{V}(\mathcal{E}) = \mathrm{Spec}(\mathbf{S}(\mathcal{E}))$ is the *vector bundle over S associated to the quasi-coherent sheaf $\mathcal{E}$.*

[It is more conventional to use the term 'vector bundle' only in the special case when $\mathcal{E}$ is locally free of finite rank.]

Note that S-morphisms $X \to \mathbf{V}(\mathcal{E})$ correspond bijectively to $\mathcal{O}_S$-algebra homomorphisms $\mathbf{S}(\mathcal{E}) \to \mathcal{A}(X)$, and in turn to $\mathcal{O}_S$-module homomorphisms $\mathcal{E} \to \mathcal{A}(X)$ [that is, the S-prescheme $\mathbf{V}(\mathcal{E})$ represents the functor $X \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{A}(X))$ from S-preschemes to sets].

(1.7.9). Taking X above to be an open subscheme $U \subseteq S$, we see that the sheaf $U \mapsto \mathrm{Hom}_S(U, \mathbf{V}(\mathcal{E}))$ of sections of the S-scheme $\mathbf{V}(\mathcal{E})$ is canonically identified with the dual $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_S)$ of $\mathcal{E}$. In particular, there is a canonical global S-section $S \to \mathbf{V}(\mathcal{E})$, the *zero section.*

(1.7.10). Now let K be a field and take $X = \mathrm{Spec}(K) = \{\xi\}$, with $f \colon X \to S$ corresponding to a field extension $k(s) \to K$ for $s \in S$, so the S-morphisms $\{\xi\} \to \mathbf{V}(\mathcal{E})$ are the *geometric points of $\mathbf{V}(\mathcal{E})$ with values in the extension K of $k(s)$*. They are identified with $\mathcal{O}_S$-module homomorphisms $\mathcal{E} \to f_*(\mathcal{O}_X)$, or equivalently with $\mathcal{O}_X$-module (*i.e.*, K-vector space) homomorphisms $f^*(\mathcal{E}) \to K$ (0, 4.4.3). By definition, $f^*(\mathcal{E}) = \mathcal{E}_s \otimes_{\mathcal{O}_s} K = \mathcal{E}^s \otimes_{k(s)} K$, where we put $\mathcal{E}^s = \mathcal{E}_s / \mathfrak{m}_s \mathcal{E}_s$. So the geometric fiber of $\mathbf{V}(\mathcal{E})$ rational over K at the point s is identified with the dual to the K-vector space $\mathcal{E}^s \otimes_{k(s)} K$, or equivalently with $(\mathcal{E}^s)^\vee \otimes_{k(s)} K$, where $(\mathcal{E}^s)$ is the dual of the $k(s)$-vector space $\mathcal{E}^s$.

Proposition (1.7.11). — *(i) $\mathbf{V}(-)$ is a contravariant functor from quasi-coherent sheaves of $\mathcal{O}_S$ modules to affine S-schemes.*

(ii) If $\mathcal{E}$ is of finite type, then $\mathbf{V}(\mathcal{E})$ is a scheme of finite type over S.

(iii) $\mathbf{V}(\mathcal{E} \oplus \mathcal{F}) = \mathbf{V}(\mathcal{E}) \times_S \mathbf{V}(\mathcal{F})$.

(iv) For any $g \colon S' \to S$, $\mathbf{V}(g^*(\mathcal{E})) \cong \mathbf{V}(\mathcal{E})_{(S')} = \mathbf{V}(\mathcal{E}) \times_S S'$.

(v) If $\mathcal{E} \to \mathcal{F}$ is surjective, then $\mathbf{V}(\mathcal{F}) \to \mathbf{V}(\mathcal{E})$ is a closed immersion.

(1.7.12). Taking $\mathcal{E} = \mathcal{O}_S$, we have $\mathbf{S}(\mathcal{E}) = \mathcal{O}_S[t]$, and $\mathbf{V}(\mathcal{E}) = S \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[t])$. We denote it $S[t]$ [or, more standardly these days, $\mathbb{A}^1_S$]. The sheaf of S-sections of $S[t]$ is identified with $\mathcal{O}_S$, by (1.7.9).

(1.7.13). For any S-prescheme X, we have $\mathrm{Hom}_S(X, S[t]) \cong \Gamma(S, \mathcal{A}(X))$, which is a ring. So the functor $\underline{S[t]}$ from S-preschemes to sets factors through commutative rings. Similarly, $\mathrm{Hom}_S(X, \mathbf{V}(\mathcal{E}))$ is a module over $\underline{S[t]}(X)$. This can be interpreted as saying that $S[t]$ is a *commutative ring scheme* over S, and $\mathbf{V}(\mathcal{E})$ is an $S[t]$-*module scheme* over S.

(1.7.14). From the structure of $S[t]$-module scheme on $\mathbf{V}(\mathcal{E})$, we can recover $\mathcal{E}$, up to canonical isomorphism. First, we recover $\mathbf{S}(\mathcal{E}) = \mathcal{A}(\mathbf{V}(\mathcal{E}))$. For any S-prescheme X, the $S[t]$-module scheme structure on $\mathbf{V}(\mathcal{E})$ identifies the the set of $\mathcal{O}_S$ *algebra* homomorphisms $\mathrm{Hom}_{\mathcal{O}_S\text{-Alg}}(\mathbf{S}(\mathcal{E}), \mathcal{A}(X))$ with $\mathcal{O}_S$ *module* homomorphisms $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{A}(X))$. In particular, this set is naturally an $\mathcal{A}(X)$-module. Now $\mathcal{E}$ is canonically identified with the sub-$\mathcal{O}_S$-module of $\mathbf{S}(\mathcal{E})$ whose sections z on an open set U have the following property: for every S-prescheme X, the evaluation map $h \to h(z)$ from $\mathrm{Hom}_{(\mathcal{O}_S|U)\text{-Alg}}(\mathbf{S}(\mathcal{E})|U, \mathcal{A}(X)|U)$ to $\Gamma(U, \mathcal{A}(X))$ is a homomorphism of $\Gamma(U, \mathcal{A}(X))$-modules.

Proposition (1.7.15). — *Let Y be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian. Every prescheme X affine and of finite type over Y is Y-isomorphic to a closed sub-Y-scheme of a Y-scheme of the form $\mathbf{V}(\mathcal{E})$, where $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type.*