9. Supplement on quasi-coherent sheaves


Proposition (9.1.1). — If $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent (resp. coherent) sheaves on a prescheme (resp. locally Noetherian prescheme) $X$, then so is $\mathcal{F} \otimes \mathcal{G}$, and it is of finite type if $\mathcal{F}$ and $\mathcal{G}$ are. If $\mathcal{F}$ is finitely presented and $\mathcal{G}$ is quasi-coherent (resp. coherent) then $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent (resp. coherent).

Definition (9.1.2). — Given $S$-preschemes $X$, $Y$ and sheaves $\mathcal{F}$, $\mathcal{G}$ of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) modules, we denote the tensor product $p_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X \times S Y}} p_2^*(\mathcal{G})$ on $X \times_S Y$ by $\mathcal{F} \otimes_S \mathcal{G}$ or $\mathcal{F} \otimes_S \mathcal{G}$.

Similar notation applies for products of more than two preschemes.

In the case $X = Y = S$, $\mathcal{F} \otimes_S \mathcal{G}$ reduces to the tensor product of $\mathcal{O}_S$ module sheaves. We have $p_1^*(\mathcal{F}) \cong \mathcal{F} \otimes_S \mathcal{O}_Y$ canonically, and likewise for $p_2$. In particular, if $Y = S$ and $f : X \rightarrow Y$ is the structure morphism of $X$ as a scheme over $Y$, then $\mathcal{O}_X \otimes_Y \mathcal{G} = f^*(\mathcal{G})$.

Thus the tensor product of sheaves on $X$ and the inverse image $f^*$ are both special cases of the general construction $\mathcal{F} \otimes_S \mathcal{G}$.

The tensor product $\mathcal{F} \otimes_S \mathcal{G}$ is a right exact covariant functor in each variable.

Proposition (9.1.3). — If $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, $Y = \text{Spec}(C)$, $\mathcal{F} = \tilde{M}$, $\mathcal{G} = \tilde{N}$, then $\mathcal{F} \times_S \mathcal{G}$ is the sheaf associated to the $B \otimes_A C$-module $M \otimes_A N$.

Proposition (9.1.4). — Given $S$-morphisms $f : T \rightarrow X$, $g : T \rightarrow Y$, we have $(f, g)^*(\mathcal{F} \otimes_S \mathcal{G}) = f^*(\mathcal{F}) \otimes_{\mathcal{O}_T} g^*(\mathcal{G})$.

Corollary (9.1.5). — Given $S$-morphisms $f : X \rightarrow X'$, $g : Y \rightarrow Y'$, we have $(f \times_S g)^*(\mathcal{F}' \otimes_S \mathcal{G}') = f^*(\mathcal{F}') \otimes_{\mathcal{O}_X} g^*(\mathcal{G}')$.

Corollary (9.1.6). — The canonical isomorphism $X \times_S Y \times_S Z \cong (X \times_S Y) \times_S Z$ identifies $\mathcal{F} \otimes_S \mathcal{G} \otimes_S \mathcal{H}$ with $\mathcal{F} \otimes_S (\mathcal{G} \otimes_S \mathcal{H})$.

Corollary (9.1.7). — The canonical isomorphism $X \times_S S \cong X$ identifies $\mathcal{F} \otimes_S \mathcal{O}_S$ with $\mathcal{F}$.

(9.1.8). Given a quasi-coherent sheaf of $\mathcal{O}_X$ modules on an $S$ prescheme $X$ and a morphism $\phi : S' \rightarrow S$ we denote by $\mathcal{F}^{(\phi)}$ or $\mathcal{F}^{(S')}$ the sheaf $\mathcal{F} \otimes_S OS'$ on $X^{(S')} = X \times_S S'$; thus $\mathcal{F}^{S'} = p^*\mathcal{F}$, where $p : X^{(S')} \rightarrow X$ is the projection.

Proposition (9.1.9). — Given $S'' \rightarrow S' \rightarrow S$, we have $(\mathcal{F}^{(\phi)})^{(\phi')}(p') = \mathcal{F}^{(\phi \circ \phi')}$.

Proposition (9.1.10). — Let $f : X \rightarrow Y$ be an $S$-morphism, $\mathcal{G}$ an $\mathcal{O}_Y$-module, and $S' \rightarrow S$ an $S$-prescheme. Then $f^{(S')}_*(\mathcal{G}^{(S')}) = (f^*(\mathcal{G}))^{(S')}$.

Corollary (9.1.11). — Given $S$-preschemes $X$, $Y$ and $S'$, the canonical isomorphism $X^{(S')} \times_{S'} Y^{(S')} \cong (X \times_S Y)^{(S')}$ identifies $\mathcal{F}^{(S')} \otimes_{S'} \mathcal{G}^{(S')}$ with $(\mathcal{F} \otimes_S \mathcal{G})^{(S')}$. 

Synopsis of material from EGA Chapter I, §9
Proposition (9.1.12). — With the notation of (9.1.2), let $z$ be a point of $X \times_S Y$, $x = p_1(z)$, $y = p_2(z)$. The stalk $(\mathcal{F} \otimes_S \mathcal{G})_z$ is isomorphic to $(\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z) \otimes_{\mathcal{O}_z} (\mathcal{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_z) = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z \otimes_{\mathcal{O}_y} \mathcal{G}_y$.

Corollary (9.1.13). — If $\mathcal{F}$, $\mathcal{G}$ are of finite type, then $\text{Supp}(\mathcal{F} \otimes_S \mathcal{G}) = p_1^{-1}(\text{Supp}(\mathcal{F})) \cap p_2^{-1}(\text{Supp}(\mathcal{G}))$.

9.2. Direct image of a quasi-coherent sheaf.

Proposition (9.2.1). — Let $f : X \to Y$ be a morphism of preschemes. Suppose $Y$ has an open affine covering $(Y_\alpha)$ such that each $f^{-1}(Y_\alpha)$ admits a finite affine covering $(X_\alpha)$, and each $X_\alpha \cap X_\beta$ admits a finite affine covering. If $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then $f_* \mathcal{O}_X$ is quasi-coherent.

Corollary (9.2.2). — The conclusion of (9.2.1) holds under any of the conditions
(a) $f$ is separated and quasi-compact,
(b) $f$ is separated and of finite type,
(c) $f$ is quasi-compact and the underlying space of $X$ is locally Noetherian.

[The hypothesis in (9.2.1) is equivalent to $f$ being quasi-compact and quasi-separated (IV, 1.7.4).]

9.3. Extending sections of quasi-coherent sheaves.

Theorem (9.3.1). — Let $X$ be a prescheme. Assume either that the underlying space of $X$ is Noetherian or that $X$ is quasi-compact and separated. Let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$ modules (0, 5.4.1), $f \in \mathcal{L}(X)$ a global section, $X_f$ the open set $\{x \in X \mid f(x) \neq 0\}$ (0, 5.5.1), and $\mathcal{F}$ a quasi-coherent sheaf.

(i) If $s \in \Gamma(X, \mathcal{F})$ has $s|_{X_f} = 0$, then $s \otimes f^n = 0$ for some $n > 0$.

(ii) For every $s \in \Gamma(X_f, \mathcal{F})$ there is an $n > 0$ such that $s \otimes f^n$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^\otimes n$.

[Remark: either of the hypotheses on $X$ implies that $X$ is quasi-compact and quasi-separated. The theorem actually holds under this more general hypothesis (IV, 1.7.5).]

Corollary (9.3.2). — In the situation of (9.3.1), consider the graded ring $A_* = \Gamma_*(\mathcal{L})$ and graded $A_*$ module $M_* = \Gamma_*(\mathcal{L}, \mathcal{F})$ (0,5.4.6). For any integer $n$ and $f \in A_n$, $\Gamma(X_f, \mathcal{F})$ is canonically isomorphic to the degree zero component $(M_*)_0$ of $(M_*)_f$, as a module over $A_0 \cong \Gamma(X_f, \mathcal{O})$.

Corollary (9.3.3). — In the situation of (9.3.1), suppose $\mathcal{L} = \mathcal{O}_X$. Setting $A = \Gamma(X, \mathcal{O}_X)$ and $M = \mathcal{F}(X)$, the $A_f$ module $\mathcal{F}(X_f)$ is canonically isomorphic to $M_f$.

Proposition (9.3.4). — Let $X$ be a Noetherian prescheme, $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$ modules on $X$, and $\mathcal{J} \subseteq \mathcal{O}_X$ a coherent ideal sheaf. If $\text{Supp}(\mathcal{F}) \subseteq \text{Supp}(\mathcal{O}_X/\mathcal{J})$, there is an $n > 0$ such that $\mathcal{J}^n \mathcal{F} = 0$.

9.4. Extending quasi-coherent sheaves.
(9.4.1). Let \( X \) be a topological space and \( j: U \to X \) the inclusion of an open subset. Let \( \mathcal{F} \) be a sheaf (of sets, groups, rings...) on \( X \) and \( \mathcal{G} \) a subsheaf of \( \mathcal{F}|_U = j^{-1}\mathcal{F} \). Then \( j_*\mathcal{G} \) is a subsheaf of \( j_*j^{-1}\mathcal{F} \), and the canonical homomorphism \( \rho: \mathcal{F} \to j_*j^{-1}\mathcal{F} \) gives us a subsheaf \( \overline{\mathcal{G}} = \rho^{-1}(\mathcal{G}) \subseteq \mathcal{F} \). For any open \( V \subseteq X \), \( \mathcal{G}(V) \) is the set of sections \( s \in \mathcal{F}(V) \) such that \( s|_{(V \cap U)} \) belongs to \( \mathcal{G}(V \cap U) \). In other words, \( \overline{\mathcal{G}} \) is the largest subsheaf of \( \mathcal{F} \) such that \( \overline{\mathcal{G}}|_U = \mathcal{G} \). The subsheaf \( \mathcal{G} \) is called the canonical extension of the subsheaf \( \mathcal{G} \subseteq \mathcal{F}|_U \) to a subsheaf of \( \mathcal{F} \).

Proposition (9.4.2). — Let \( X \) be a prescheme and \( U \subseteq X \) an open subset such that the inclusion \( j: U \to X \) is a quasi-compact morphism.

(i) For every quasi-coherent sheaf \( \mathcal{G} \) of \( \mathcal{O}_X|U \) modules, \( j_*\mathcal{G} \) is quasi-coherent and \( \mathcal{G} = j^{-1}j_*(\mathcal{G})|_U \).

(ii) For every quasi-coherent sheaf \( \mathcal{F} \) of \( \mathcal{O}_X \) modules and quasi-coherent submodule sheaf \( \mathcal{G} \subseteq \mathcal{F}|_U \), the canonical extension \( \overline{\mathcal{G}} \subseteq \mathcal{F} \) is quasi-coherent.

The hypothesis that \( j: U \to X \) is quasi-compact holds automatically either if the underlying space of \( X \) is locally Noetherian (6.6.4 (i)), or if \( U \) is quasi-compact and \( X \) is separated, by (5.5.6) [this can be weakened to \( U \) quasi-compact and \( X \) quasi-separated by (IV, 1.2.7)].

Corollary (9.4.3). — Let \( X \) be a prescheme and \( U \) a quasi-compact open subset such that \( j: U \to X \) is quasi-compact [for instance, if \( X \) is quasi-separated]. Suppose further that every quasi-coherent sheaf of \( \mathcal{O}_X \) modules is a direct limit of subsheaves of \( \mathcal{O}_X \) modules of finite type (for instance, if \( X \) is affine). Let \( \mathcal{F} \) be a quasi-coherent sheaf of \( \mathcal{O}_X \) modules and \( \mathcal{G} \subseteq \mathcal{F}|_U \) a quasi-coherent submodule sheaf of finite type. Then there exists a quasi-coherent submodule sheaf \( \mathcal{G}' \subseteq \mathcal{F} \) of \( \mathcal{O}_X \) modules of finite type such that \( \mathcal{G}'|_U = \mathcal{G} \).

Remark (9.4.4). — Suppose \( X \) has the property that the inclusion \( U \to X \) is quasi-compact for every open \( U \subseteq X \). Then the hypothesis in (9.4.3) that every quasi-coherent sheaf of \( \mathcal{O}_X \) modules is a direct limit of subsheaves of \( \mathcal{O}_X \) modules of finite type is valid if the conclusion of (9.4.3) holds for every affine open \( U \subseteq X \), quasi-coherent sheaf \( \mathcal{F} \) of \( \mathcal{O}_X \) modules, and quasi-coherent submodule sheaf \( \mathcal{G} \subseteq \mathcal{F}|_U \) of finite type.

Corollary (9.4.5). — Under the hypotheses of (9.4.3), every quasi-coherent sheaf \( \mathcal{G} \) of \( \mathcal{O}_X|U \) modules of finite type is the restriction \( \mathcal{G} = \mathcal{G}'|_U \) of a quasi-coherent sheaf of \( \mathcal{O}_X \) modules of finite type.

Lemma (9.4.6). — Let \( X \) be a prescheme, \( (V_\lambda)_{\lambda \in L} \) an open affine covering of \( X \) indexed by a well-ordered set \( L \), and \( U \subseteq X \) an open subset. For each \( \lambda \in L \), put \( W_\lambda = \bigcup_{\mu < \lambda} V_\mu \). Suppose that (i) for each \( \lambda \in L \), \( V_\lambda \cap W_\lambda \) is quasi-compact and (ii) the immersion morphism \( U \to X \) is quasi-compact. Then for every quasi-coherent sheaf \( \mathcal{F} \) of \( \mathcal{O}_X \) modules and quasi-coherent submodule sheaf \( \mathcal{G} \subseteq \mathcal{F}|_U \) of finite type, there exists a quasi-coherent submodule sheaf \( \mathcal{G}' \subseteq \mathcal{F} \) of finite type such that \( \mathcal{G} = \mathcal{G}'|_U \).

Theorem (9.4.7). — Let \( U \) be an open subset of a prescheme \( X \). Suppose that either of the following conditions holds:

(a) the underlying space of \( X \) is locally Noetherian, or
(b) $X$ is separated and quasi-compact and $U$ is quasi-compact.

Then for every quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules and quasi-coherent submodule sheaf $\mathcal{G} \subseteq \mathcal{F}|_U$ of finite type, there exists a quasi-coherent submodule sheaf $\mathcal{G}' \subseteq \mathcal{F}$ of finite type such that $\mathcal{G} = \mathcal{G}'|_U$.

Corollary (9.4.8). — Under the hypotheses of (9.4.7), every quasi-coherent sheaf $\mathcal{G}$ of $\mathcal{O}_X|_U$ modules of finite type is the restriction $\mathcal{G} = \mathcal{G}'|_U$ of a quasi-coherent sheaf $\mathcal{G}$ of $\mathcal{O}_X$ modules of finite type.

Corollary (9.4.9). — If the underlying space of $X$ is locally Noetherian, or if $X$ is separated and quasi-compact, then every quasi-coherent sheaf of $\mathcal{O}_X$ modules is a direct limit of submodule sheaves of finite type.

Corollary (9.4.10). — Under the hypotheses of (9.4.9), if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$ modules such that every quasi-coherent submodule sheaf of finite type of $\mathcal{F}$ is generated by global sections, then $\mathcal{F}$ is generated by global sections.

[In (9.4.7 (b)) and the subsequent corollaries one can weaken ‘separated’ to ‘quasi-separated’ (IV, 1.7.7).]

9.5. Closed image of a prescheme; closure of a sub-prescheme.

Proposition (9.5.1). — Let $f : X \to Y$ be a morphism of preschemes such that $f_*\mathcal{O}_X$ is quasi-coherent (which holds if $f$ is quasi-compact, and either $f$ is separated or the underlying space of $X$ is locally Noetherian or more generally if $f$ is quasi-compact and quasi-separated). Then there is a smallest closed sub-prescheme $Y' \subseteq Y$ such that $f$ factors through the inclusion $j : Y' \to Y$, or equivalently (4.4.1), such that the sub-prescheme $f^{-1}(Y') \subseteq X$ is equal to $X$.

Corollary (9.5.2). — More precisely, the kernel $\mathcal{I}$ of $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is quasi-coherent and the closed sub-prescheme $Y'$ defined by $\mathcal{I}$ has the property in (9.5.1).

Definition (9.5.3). — $Y'$ with the property in (9.5.1) is called the closed image of $X$ under the morphism $f$.

[Remark: the closed image $Y'$ actually exists for every morphism $f$. Since a sum of quasi-coherent ideal sheaves is quasi-coherent (4.1.1), every ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ contains a unique largest quasi-coherent ideal sheaf $\mathcal{I}'$. If we take $\mathcal{I} = \ker(f^*)$, then the closed subscheme $Y'$ defined by $\mathcal{I}'$ has the property in (9.5.1), even if $\mathcal{I}$ is not quasi-coherent. Moreover, for $\mathcal{I}$ to be quasi-coherent it is not necessary that $f_*\mathcal{O}_X$ be quasi-coherent. For instance, it suffices that $f$ be quasi-compact.]

Proposition (9.5.4). — If $f_*\mathcal{O}_X$ is quasi-coherent, then the underlying space of $Y'$ is the closure of $f(X)$ in $X$.

[The weaker condition that $\ker(f^*)$ be quasi-coherent suffices.]
Proposition (9.5.5). — (transitivity of closed image) Given $X \to Y \to Z$, let $Y'$ be the closed image of $X$ under $f$. Then the closed image of $X$ under $g \circ f$ is equal to the closed image of $Y'$ under the restriction $g': Y' \to Z$ of $g$.

Corollary (9.5.6). — Let $f: X \to Y$ be an $S$-morphism such that the closed image of $X$ under $f$ is equal to $Y$. Let $Z$ be a separated prescheme over $S$. If $g_1, g_2$ are morphisms $Y \to Z$ such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.

Remark (9.5.7). — If we also suppose $X, Y$ separated, the conclusion of (9.5.6) says that $f$ is an epimorphism in the category of separated preschemes over $S$.

Proposition (9.5.8). — Under the hypotheses of (9.5.1), if $V \subseteq Y$ is open, then $V \cap Y'$ is the closed image of $f^{-1}(V)$ under $f|f^{-1}(V)$.

[Again the weaker condition that $\text{ker}(f^\circ)$ be quasi-coherent suffices.]

Proposition (9.5.9). — Let $Y'$ be the closed image of $X$ under $f: X \to Y$.

(i) If $X$ is reduced, then so is $Y'$.

(ii) If $f_*\mathcal{O}_X$ is quasi-coherent and $X$ is irreducible (resp. integral), then so is $Y'$.

[In (ii), the weaker condition that $\text{ker}(f^\circ)$ be quasi-coherent suffices.]

Proposition (9.5.10). — Let $Y$ be a sub-prescheme of $X$ such that the inclusion $i: Y \to X$ is quasi-compact. Then there is a smallest closed subscheme $\overline{Y}$ majorizing $Y$, its underlying space is the closure of $Y$, $Y$ is open in its closure, and $Y$ is equal to the restriction of $\overline{Y}$ to this open subset.

Corollary (9.5.11). — Under the hypotheses of (9.5.10), if the restriction of a section $s \in \mathcal{O}_{\overline{Y}}(V)$ to $V \cap Y$ is zero, then $s = 0$.

9.6. Quasi-coherent sheaves of algebras; change of stucture sheaf.

Proposition (9.6.1). — Let $X$ be a prescheme, $\mathcal{B}$ a quasi-coherent sheaf of $\mathcal{O}_X$-algebras (0, 5.1.3). A $\mathcal{B}$-module sheaf $\mathcal{F}$ is quasi-coherent as a sheaf of modules on the ringed space $(X, \mathcal{B})$ if and only if $\mathcal{F}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$ modules.

[This result, which is proved by reduction to affines, is specific to preschemes and does not hold on a general ringed space.]

(9.6.2). A quasi-coherent sheaf of $\mathcal{O}_X$ algebras $\mathcal{B}$ is of finite type if every $x \in X$ has an affine neighborhood $U = \text{Spec}(A)$ such that $B = \mathcal{B}(U)$ is a finitely-generated $A$-algebra. Then the same thing holds on $U_f = \text{Spec}(A_f)$ for $f \in A$. It follows that if $\mathcal{B}$ is of finite type, then $\mathcal{B}|V$ is of finite type for every open $V \subseteq X$.

Proposition (9.6.3). — If $X$ is locally Noetherian, then every $\mathcal{O}_X$-algebra $\mathcal{B}$ of finite type is a coherent sheaf of rings (0, 5.3.7).

Corollary (9.6.4). — Under the hypotheses of (9.6.3) a sheaf of $\mathcal{B}$ modules $\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is of finite type as a sheaf of $\mathcal{B}$ modules and quasi-coherent as a sheaf of $\mathcal{O}_X$ modules.
modules. In this case, if \( \mathcal{G} \) is a submodule sheaf or quotient sheaf of \( \mathcal{F} \), then \( \mathcal{G} \) is a coherent sheaf of \( \mathcal{B} \) modules if and only if \( \mathcal{G} \) is quasi-coherent as a sheaf of \( \mathcal{O}_X \) modules.

**Proposition (9.6.5).** — Let \( X \) be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian. Then every quasi-coherent \( \mathcal{O}_X \)-algebra \( \mathcal{B} \) of finite type contains an \( \mathcal{O}_X \)-submodule of finite type which generates \( \mathcal{B} \) as an \( \mathcal{O}_X \)-algebra.

[The proposition holds if \( X \) is quasi-compact and quasi-separated (IV, 1.7.9), a condition weaker than each of the two specified hypotheses.]

**Proposition (9.6.6).** — Let \( X \) be a quasi-compact scheme, or a prescheme whose underlying space is locally Noetherian. Then every quasi-coherent sheaf of \( \mathcal{O}_X \) algebras \( \mathcal{B} \) is the inductive limit of its quasi-coherent subalgebra sheaves of finite type.

[One can weaken ‘quasi-compact scheme’ to ‘quasi-compact and quasi-separated prescheme’ (IV, 1.7.9).]