Synopsis of material from EGA Chapter I, §9

9. Supplement on quasi-coherent sheaves

9.1. **Tensor product of quasi-coherent sheaves.**

Proposition (9.1.1). — *If $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent (resp. coherent) sheaves on a prescheme (resp. locallly Noetherian prescheme) X, then so is $\mathcal{F} \otimes \mathcal{G}$, and it is of finite type if $\mathcal{F}$ and $\mathcal{G}$ are. If $\mathcal{F}$ is finitely presented and $\mathcal{G}$ is quasi-coherent (resp. coherent) then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent (resp. coherent).*

Definition (9.1.2). — Given S-preschemes X, Y and sheaves $\mathcal{F}$, $\mathcal{G}$ of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) modules, we denote the tensor product $p_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X\times_S Y}} p_2^*(\mathcal{G})$ on $X \times_S Y$ by $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ or $\mathcal{F} \otimes_S \mathcal{G}$.

Similar notation applies for products of more than two preschemes.

In the case $X = Y = S$, $\mathcal{F} \otimes_S \mathcal{G}$ reduces to the tensor product of $\mathcal{O}_S$ module sheaves. We have $p_1^*(\mathcal{F}) \cong \mathcal{F} \otimes_S \mathcal{O}_Y$ canonically, and likewise for p_2. In particular, if $Y = S$ and $f\colon X \to Y$ is the structure morphism of X as a scheme over Y, then $\mathcal{O}_X \otimes_Y \mathcal{G} = f^*(\mathcal{G})$. Thus the tensor product of sheaves on X and the inverse image f^* are both special cases of the general construction $\mathcal{F} \otimes_S \mathcal{G}$.

The tensor product $\mathcal{F} \otimes_S \mathcal{G}$ is a right exact covariant functor in each variable.

Proposition (9.1.3). — *If $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(C)$, $\mathcal{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$, then $\mathcal{F} \times_S \mathcal{G}$ is the sheaf associated to the $B \otimes_A C$-module $M \otimes_A N$.*

Proposition (9.1.4). — *Given S-morphisms $f\colon T \to X$, $g\colon T \to Y$, we have $(f, g)^*(\mathcal{F} \otimes_S \mathcal{G}) = f^*(\mathcal{F}) \otimes_{\mathcal{O}_T} g^*(\mathcal{G})$.*

Corollary (9.1.5). — *Given S-morphisms $f\colon X \to X'$, $g\colon Y \to Y'$, we have $(f \times_S g)^*(\mathcal{F}' \otimes_S \mathcal{G}') = f^*(\mathcal{F}') \otimes_S g^*(\mathcal{G}')$.*

Corollary (9.1.6). — *The canonical isomorphism $X \times_S Y \times_S Z \cong (X \times_S Y) \times_S Z$ identifies $\mathcal{F} \otimes_S \mathcal{G} \otimes_S \mathcal{H}$ with $(\mathcal{F} \otimes_S \mathcal{G}) \otimes_S \mathcal{H}$.*

Corollary (9.1.7). — *The canonical isomorphism $X \times_S S \cong X$ identifies $\mathcal{F} \otimes_S \mathcal{O}_S$ with $\mathcal{F}$.*

(9.1.8). Given a quasi-coherent sheaf of $\mathcal{O}_X$ modules on an S prescheme X and a morphism $\phi\colon S' \to S$ we denote by $\mathcal{F}_{(\phi)}$ or $\mathcal{F}_{(S')}$ the sheaf $\mathcal{F} \otimes_S \mathcal{O}S'$ on $X_{(S')} = X \times_S S'$; thus $\mathcal{F}_{S'} = p^*\mathcal{F}$, where $p\colon X_{(S')} \to X$ is the projection.

Proposition (9.1.9). — *Given $S'' \underset{\phi'}{\to} S' \underset{\phi}{\to} S$, we have $(\mathcal{F}_{(\phi)})_{(\phi')} = \mathcal{F}_{(\phi\circ\phi')}$.*

Proposition (9.1.10). — *Let $f\colon X \to Y$ be an S-morphism, $\mathcal{G}$ an $\mathcal{O}_Y$-module, and $S' \to S$ an S-prescheme. Then ${f_{(S')}}^*(\mathcal{G}_{(S')}) = (f^*(\mathcal{G}))_{(S')}$.*

Corollary (9.1.11). — *Given S-preschemes X, Y and S', the canonical isomorphism $X_{(S')} \times_{S'} Y_{(S')} \cong (X \times_S Y)_{(S')}$ identifies $\mathcal{F}_{(S')} \otimes_{S'} \mathcal{G}_{(S')}$ with $(\mathcal{F} \otimes_S \mathcal{G})_{(S')}$.*

Proposition (9.1.12). — *With the notation of (9.1.2), let z be a point of $X \times_S Y$, $x = p_1(z)$, $y = p_2(z)$. The stalk $(\mathcal{F} \otimes_S \mathcal{G})_z$ is isomorphic to $(\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z) \otimes_{\mathcal{O}_z} (\mathcal{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_z) = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z \otimes_{\mathcal{O}_y} \mathcal{G}_y$.*

Corollary (9.1.13). — *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, then* $\operatorname{Supp}(\mathcal{F} \otimes_S \mathcal{G}) = p_1^{-1}(\operatorname{Supp}(\mathcal{F})) \cap p_2^{-1}(\operatorname{Supp}(\mathcal{G}))$.

9.2. **Direct image of a quasi-coherent sheaf.**

Proposition (9.2.1). — *Let $f\colon X \to Y$ be a morphism of preschemes. Suppose Y has an open affine covering (Y_α) such that each $f^{-1}(Y_\alpha)$ admits a finite affine covering (X_{α_i}), and each $X_{\alpha i} \cap X_{\alpha j}$ admits a finite affine covering. If $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then $f_*\mathcal{O}_X$ is quasi-coherent.*

Corollary (9.2.2). — *The conclusion of (9.2.1) holds under any of the conditions*
(a) f is separated and quasi-compact,
(b) f is separated and of finite type,
(c) f is quasi-compact and the underlying space of X is locally Noetherian.

[The hypothesis in (9.2.1) is equivalent to f being *quasi-compact* and *quasi-separated* (IV, 1.7.4).]

9.3. **Extending sections of quasi-coherent sheaves.**

Theorem (9.3.1). — *Let X be a prescheme. Assume either that the underlying space of X is Noetherian or that X is quasi-compact and separated. Let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$ modules (0, 5.4.1), $f \in \mathcal{L}(X)$ a global section, X_f the open set $\{x \in X \mid f(x) \neq 0\}$ (0, 5.5.1), and $\mathcal{F}$ a quasi-coherent sheaf.*
(i) If $s \in \Gamma(X, \mathcal{F})$ has $s|X_f = 0$, then $s \otimes f^n = 0$ for some $n > 0$.
(ii) For every $s \in \Gamma(X_f, \mathcal{F})$ there is an $n > 0$ such that $s \otimes f^n$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.

[Remark: either of the hypotheses on X implies that X is quasi-compact and quasi-separated. The theorem actually holds under this more general hypothesis (IV, 1.7.5).]

Corollary (9.3.2). — *In the situation of (9.3.1), consider the graded ring $A_* = \Gamma_*(\mathcal{L})$ and graded A_* module $M_* = \Gamma_*(\mathcal{L}, \mathcal{F})$ (0,5.4.6). For any integer n and $f \in A_n$, $\Gamma(X_f, \mathcal{F})$ is canonically isomorphic to the degree zero component $((M_*)_f)_0$ of $(M_*)_f$, as a module over $A_0 \cong \Gamma(X_f, \mathcal{O})$.*

Corollary (9.3.3). — *In the situation of (9.3.1), suppose $\mathcal{L} = \mathcal{O}_X$. Setting $A = \Gamma(X, \mathcal{O}_X)$ and $M = \mathcal{F}(X)$, the A_f module $\mathcal{F}(X_f)$ is canonically isomorphic to M_f.*

Proposition (9.3.4). — *Let X be a Noetherian prescheme, $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$ modules on X, and $\mathcal{J} \subseteq \mathcal{O}_X$ a coherent ideal sheaf. If Supp$(\mathcal{F}) \subseteq \operatorname{Supp}(\mathcal{O}_X/\mathcal{J})$, there is an $n > 0$ such that $\mathcal{J}^n\mathcal{F} = 0$.*

9.4. **Extending quasi-coherent sheaves.**

(9.4.1). Let X be a topological space and $j\colon U \to X$ the inclusion of an open subset. Let $\mathcal{F}$ be a sheaf (of sets, groups, rings...) on X and $\mathcal{G}$ a subsheaf of $\mathcal{F}|U = j^{-1}\mathcal{F}$. Then $j_*\mathcal{G}$ is a subsheaf of $j_*j^{-1}\mathcal{F}$, and the canonical homomorphism $\rho\colon \mathcal{F} \to j_*j^{-1}\mathcal{F}$ gives us a subsheaf $\overline{\mathcal{G}} = \rho^{-1}(\mathcal{G}) \subseteq \mathcal{F}$. For any open $V \subseteq X$, $\overline{\mathcal{G}}(V)$ is the set of sections $s \in \mathcal{F}(V)$ such that $s|(V \cap U)$ belongs to $\mathcal{G}(V \cap U)$. In other words, $\overline{\mathcal{G}}$ is the largest subsheaf of $\mathcal{F}$ such that $\overline{\mathcal{G}}|U = \mathcal{G}$. The subsheaf $\mathcal{G}$ is called the *canonical extension* of the subsheaf $\mathcal{G} \subseteq \mathcal{F}|U$ to a subsheaf of $\mathcal{F}$.

Proposition (9.4.2). — *Let X be a prescheme and $U \subseteq X$ an open subset such that the inclusion $j\colon U \to X$ is a quasi-compact morphism.*
(i) For every quasi-coherent sheaf $\mathcal{G}$ of $\mathcal{O}_X|U$ modules, $j_*\mathcal{G}$ is quasi-coherent and $\mathcal{G} = j^{-1}j_*(\mathcal{G}) = j_*(\mathcal{G})|U$.
(ii) For every quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules and quasi-coherent submodule sheaf $\mathcal{G} \subseteq \mathcal{F}|U$, the canonical extension $\overline{\mathcal{G}} \subseteq \mathcal{F}$ is quasi-coherent.

The hypothesis that $j\colon U \to X$ is quasi-compact holds automatically either if the underlying space of X is locally Noetheian (6.6.4 (i)), or if U is quasi-compact and X is separated, by (5.5.6) [this can be weakened to U quasi-compact and X quasi-separated by (IV, 1.2.7)].

Corollary (9.4.3). — *Let X be a prescheme and U a quasi-compact open subset such that $j\colon U \to X$ is quasi-compact [for instance, if X is quasi-separated]. Suppose further that every quasi-coherent sheaf of $\mathcal{O}_X$ modules is a direct limit of subsheaves of $\mathcal{O}_X$ modules of finite type (for instance, if X is affine). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$ modules and $\mathcal{G} \subseteq \mathcal{F}|U$ a quasi-coherent submodule sheaf of finite type. Then there exists a quasi-coherent submodule sheaf $\mathcal{G}' \subseteq \mathcal{F}$ of $\mathcal{O}_X$ modules of finite type such that $\mathcal{G}'|U = \mathcal{G}$.*

Remark (9.4.4). — Suppose X has the property that the inclusion $U \to X$ is quasi-compact for every open $U \subseteq X$. Then the hypothesis in (9.4.3) that every quasi-coherent sheaf of $\mathcal{O}_X$ modules is a direct limit of subsheaves of $\mathcal{O}_X$ modules of finite type is valid if the conclusion of (9.4.3) holds for every affine open $U \subseteq X$, quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules, and quasi-coherent submodule sheaf $\mathcal{G} \subseteq \mathcal{F}|U$ of finite type.

Corollary (9.4.5). — *Under the hypotheses of (9.4.3), every quasi-coherent sheaf $\mathcal{G}$ of $\mathcal{O}_X|U$ modules of finite type is the restriction $\mathcal{G} = \mathcal{G}'|U$ of a quasi-coherent sheaf of $\mathcal{O}_X$ modules of finite type.*

Lemma (9.4.6). — *Let X be a prescheme, $(V_\lambda)_{\lambda \in L}$ an open affine covering of X indexed by a well-ordered set L, and $U \subseteq X$ an open subset. For each $\lambda \in L$, put $W_\lambda = \bigcup_{\mu<\lambda} V_\mu$. Suppose that (i) for each $\lambda \in L$, $V_\lambda \cap W_\lambda$ is quasi-compact and (ii) the immersion morphism $U \to X$ is quasi-compact. Then for every quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules and quasi-coherent submodule sheaf $\mathcal{G} \subseteq \mathcal{F}|U$ of finite type, there exists a quasi-coherent submodule sheaf $\mathcal{G}' \subseteq \mathcal{F}$ of finite type such that $\mathcal{G} = \mathcal{G}'|U$.*

Theorem (9.4.7). — *Let U be an open subset of a prescheme X. Suppose that either of the following conditions holds:*
(a) the underlying space of X is locally Noetherian, or

(b) X is separated and quasi-compact and U is quasi-compact.

Then for every quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules and quasi-coherent submodule sheaf $\mathcal{G} \subseteq \mathcal{F}|U$ of finite type, there exists a quasi-coherent submodule sheaf $\mathcal{G}' \subseteq \mathcal{F}$ of finite type such that $\mathcal{G} = \mathcal{G}'|U$.

Corollary (9.4.8). — *Under the hypotheses of (9.4.7), every quasi-coherent sheaf $\mathcal{G}$ of $\mathcal{O}_X|U$ modules of finite type is the restriction $\mathcal{G} = \mathcal{G}'|U$ of a quasi-coherent sheaf $\mathcal{G}$ of $\mathcal{O}_X$ modules of finite type.*

Corollary (9.4.9). — *If the underlying space of X is locally Noetherian, or if X is separated and quasi-compact, then every quasi-coherent sheaf of $\mathcal{O}_X$ modules is a direct limit of submodule sheaves of finite type.*

Corollary (9.4.10). — *Under the hypotheses of (9.4.9), if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$ modules such that every quasi-coherent submodule sheaf of finite type of $\mathcal{F}$ is generated by global sections, then $\mathcal{F}$ is generated by global sections.*

[In (9.4.7 (b)) and the subsequent corollaries one can weaken ‘separated’ to ‘quasi-separated’ (IV, 1.7.7).]

9.5. **Closed image of a prescheme; closure of a sub-prescheme.**

Proposition (9.5.1). — *Let $f\colon X \to Y$ be a morphism of preschemes such that $f_*\mathcal{O}_X$ is quasi-coherent (which holds if f is quasi-compact, and either f is separated or the underlying space of X is locally Noetherian [or more generally if f is quasi-compact and quasi-separated]). Then there is a smallest closed sub-prescheme $Y' \subseteq Y$ such that f factors through the inclusion $j\colon Y' \to Y$, or equivalently (4.4.1), such that the sub-prescheme $f^{-1}(Y') \subseteq X$ is equal to X.*

Corollary (9.5.2). — *More precisely, the kernel $\mathcal{I}$ of $f^\flat\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is quasi-coherent and the closed sub-prescheme Y' defined by $\mathcal{I}$ has the property in (9.5.1).*

Definition (9.5.3). — Y' with the property in (9.5.1) is called the *closed image* of X under the morphism f.

[Remark: the closed image Y' actually exists for every morphism f. Since a sum of quasi-coherent ideal sheaves is quasi-coherent (4.1.1), every ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ contains a unique largest quasi-coherent ideal sheaf $\mathcal{I}'$. If we take $\mathcal{I} = \ker(f^\flat)$, then the closed subscheme Y' defined by $\mathcal{I}'$ has the property in (9.5.1), even if $\mathcal{I}$ is not quasi-coherent. Moreover, for $\mathcal{I}$ to be quasi-coherent it is not necessary that $f_*\mathcal{O}_X$ be quasi-coherent. For instance, it suffices that f be quasi-compact.]

Proposition (9.5.4). — *If $f_*\mathcal{O}_X$ is quasi-coherent, then the underlying space of Y' is the closure of $f(X)$ in X.*

[The weaker condition that $\ker(f^\flat)$ be quasi-coherent suffices.]

Proposition (9.5.5). — (transitivity of closed image) *Given* $X \underset{f}{\to} Y \underset{g}{\to} Z$, *let* Y' *be the closed image of* X *under* f. *Then the closed image of* X *under* $g \circ f$ *is equal to the closed image of* Y' *under the restriction* $g' \colon Y' \to Z$ *of* g.

Corollary (9.5.6). — *Let* $f \colon X \to Y$ *be an* S*-morphism such that the closed image of* X *under* f *is equal to* Y. *Let* Z *be a separated prescheme over* S. *If* g_1, g_2 *are morphisms* $Y \to Z$ *such that* $g_1 \circ f = g_2 \circ f$, *then* $g_1 = g_2$.

Remark (9.5.7). — If we also suppose X, Y separated, the conclusion of (9.5.6) says that f is an epimorphism in the category of separated preschemes over S.

Proposition (9.5.8). — *Under the hypotheses of (9.5.1), if* $V \subseteq Y$ *is open, then* $V \cap Y'$ *is the closed image of* $f^{-1}(V)$ *under* $f|f^{-1}(V)$.

[Again the weaker condition that $\ker(f^\flat)$ be quasi-coherent suffices.]

Proposition (9.5.9). — *Let* Y' *be the closed image of* X *under* $f \colon X \to Y$.
(i) If X *is reduced, then so is* Y'.
(ii) If $f_*\mathcal{O}_X$ *is quasi-coherent and* X *is irreducible (resp. integral), then so is* Y'.

[In (ii), the weaker condition that $\ker(f^\flat)$ be quasi-coherent suffices.]

Proposition (9.5.10). — *Let* Y *be a sub-prescheme of* X *such that the inclusion* $i \colon Y \to X$ *is quasi-compact. Then there is a smallest closed subscheme* $\overline{Y}$ *majorizing* Y, *its underlying space is the closure of* Y, Y *is open in its closure, and* Y *is equal to the restriction of* $\overline{Y}$ *to this open subset.*

Corollary (9.5.11). — *Under the hypotheses of (9.5.10), if the restriction of a section* $s \in \mathcal{O}_{\overline{Y}}(V)$ *to* $V \cap Y$ *is zero, then* $s = 0$.

9.6. **Quasi-coherent sheaves of algebras; change of stucture sheaf.**

Proposition (9.6.1). — *Let* X *be a prescheme,* $\mathcal{B}$ *a quasi-coherent sheaf of* $\mathcal{O}_X$*-algebras (0, 5.1.3). A* $\mathcal{B}$*-module sheaf* $\mathcal{F}$ *is quasi-coherent as a sheaf of modules on the ringed space* $(X, \mathcal{B})$ *if and only if* $\mathcal{F}$ *is quasi-coherent as a sheaf of* $\mathcal{O}_X$ *modules.*

[This result, which is proved by reduction to affines, is specific to preschemes and does not hold on a general ringed space.]

(9.6.2). A quasi-coherent sheaf of $\mathcal{O}_X$ algebras $\mathcal{B}$ is *of finite type* if every $x \in X$ has an affine neighborhood $U = \mathrm{Spec}(A)$ such that $B = \mathcal{B}(U)$ is a finitely-generated A-algebra. Then the same thing holds on $U_f = \mathrm{Spec}(A_f)$ for $f \in A$. It follows that if $\mathcal{B}$ is of finite type, then $\mathcal{B}|V$ is of finite type for every open $V \subseteq X$.

Proposition (9.6.3). — *If* X *is locally Noetherian, then every* $\mathcal{O}_X$*-algebra* $\mathcal{B}$ *of finite type is a coherent sheaf of rings (0, 5.3.7).*

Corollary (9.6.4). — *Under the hypotheses of (9.6.3) a sheaf of* $\mathcal{B}$ *modules* $\mathcal{F}$ *is coherent if and only if* $\mathcal{F}$ *is of finite type as a sheaf of* $\mathcal{B}$ *modules and quasi-coherent as a sheaf of* $\mathcal{O}_X$

modules. In this case, if $\mathcal{G}$ is a submodule sheaf or quotient sheaf of $\mathcal{F}$, then $\mathcal{G}$ is a coherent sheaf of $\mathcal{B}$ modules if and only if $\mathcal{G}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$ modules.

Proposition (9.6.5). — *Let X be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian. Then every quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ of finite type contains an $\mathcal{O}_X$-submodule of finite type which generates $\mathcal{B}$ as an $\mathcal{O}_X$-algebra.*

[The proposition holds if X is quasi-compact and quasi-separated (IV, 1.7.9), a condition weaker than each of the two specified hypotheses.]

Proposition (9.6.6). — *Let X be a quasi-compact scheme, or a prescheme whose underlying space is locally Noetherian. Then every quasi-coherent sheaf of $\mathcal{O}_X$ algebras $\mathcal{B}$ is the inductive limit of its quasi-coherent subalgebra sheaves of finite type.*

[One can weaken 'quasi-compact scheme' to 'quasi-compact and quasi-separated prescheme' (IV, 1.7.9).]