Synopsis of material from EGA Chapter I, §6

6. Finiteness conditions

6.1. **Noetherian and locally Noetherian preschemes.**

Definition (6.1.1). — X is *locally Noetherian* if it has a covering by open affines $\mathrm{Spec}(R)$ with R Noetherian. X is *Noetherian* if it has a finite such covering [Liu, 2.3.45].

If X is locally Noetherian, then $\mathcal{O}_X$ is coherent, a quasi-coherent sheaf of $\mathcal{O}_X$ modules is coherent iff it is locally finitely generated, and every quasi-coherent subsheaf of a coherent sheaf of $\mathcal{O}_X$ modules is coherent.

Proposition (6.1.2). — *X is Noetherian iff it is locally Noetherian and quasi-compact; then its underlying space is a Noetherian topological space (but not conversely).*

Proposition (6.1.3). — *The following are equivalent [Liu, Ex. 2.3.16]:*
(a) $\mathrm{Spec}(A)$ *is Noetherian*
(b) $\mathrm{Spec}(A)$ *is locally Noetherian*
(c) A is Noetherian.

Proposition (6.1.4). — *Any open or closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian [Liu, 2.3.46].*

(6.1.5). Since the tensor product of Noetherian algebras is not necessarily Noetherian, the product of two Noetherian schemes over a scheme S is not necessarily Noetherian.

Proposition (6.1.6). — *If X is Noetherian, the nilradical $\mathcal{N}_X$ of $\mathcal{O}_X$ is nilpotent.*

Corollary (6.1.7). — *If X is Noetherian, then X is affine iff X_{red} is.*

Lemma (6.1.8). — *Let X be a topological space. Suppose $x \in X$ has an open neighborhood with finitely many irreducible components. Then x has an open neighborhood V such that every open $W \subseteq V$ containing x is connected.*

Corollary (6.1.9). — *A locally Noetherian topological space is locally connected, which implies that its connected components are open.*

Proposition (6.1.10). — *If X is a locally Noetherian topological space, the following are equivalent.*
(a) The irreducible components of X are open.
(b) The irreducible components of X are the same as its connected components.
(c) The connected components of X are irreducible.
(d) Distinct irreducible components of X are disjoint.
If X is a prescheme, the above are also equivalent to:
(e) For every $x \in X$, $\mathrm{Spec}(\mathcal{O}_{X,x})$ *is irreducible, that is, the nilradical of $\mathcal{O}_{X,x}$ is prime.*

Corollary (6.1.11). — *Let X be a locally Noetherian space. Then X is irreducible if and only it is connected and non-empty, and its distinct irreducible components are disjoint. If X is a prescheme, the last condition is equivalent to* $\mathrm{Spec}(\mathcal{O}_{X,x})$ *being irreducible for all $x \in X$.*

Corollary (6.1.12). — *Let X be a locally Noetherian prescheme. Then X is integral iff X is connected and $\mathcal{O}_{X,x}$ is an integral domain for all $x \in X$ [Liu, Ex. 4.4.4].*

Proposition (6.1.13). — *If X is a locally Noetherian prescheme, and $x \in X$ is such that the nilradical $\mathcal{N}_x$ of $\mathcal{O}_{X,x}$ is prime (resp. such that $\mathcal{O}_{X,x}$ is reduced; is a domain), then x has a neighborhood U which is irreducible (resp. reduced; integral) [Liu, Ex. 2.4.9].*

6.2. **Artinian preschemes.**

Definition (6.2.1). — A prescheme is *Artinian* if it is affine and its ring is Artinian.

Proposition (6.2.2). — *The following properties of a prescheme X are equivalent:*
(a) X is Artinian.
(b) X is Noetherian and its underlying space is discrete.
(c) X is Noetherian and every point of X is closed (X is a T_1 space).
When the above hold, the underlying space of X is finite, and the ring A of X is the direct product of the (Artinian) local rings of the points of X.

6.3. **Morphisms of finite type.**

Definition (6.3.1). — A morphism $f\colon X \to Y$ is *of finite type* if Y can be covered by open affine subsets $V \cong \operatorname{Spec}(A)$ satisfying the property
(P): $f^{-1}(V)$ is a finite union of affine opens $U_i \cong \operatorname{Spec}(R_i)$ for which R_i is finitely generated as an A algebra.

One also says that X is a prescheme of finite type over Y, or a Y-prescheme of finite type.

[Liu (Def. 3.2.1) uses a different definition, equivalent to the above by Prop. 3.2.2 in Liu, plus (6.3.3), (6.6.3) and the fact that for a morphism to be locally of finite type is a local property on both X and Y—see (6.6.2).]

Proposition (6.3.2). — *If $f\colon X \to Y$ is of finite type, then property (P) holds for every open affine $V \subseteq Y$.*

This implies that the property that f is of finite type is *local on Y*.

Proposition (6.3.3). — *A morphism of affine schemes* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is of finite type if and only if B is a finitely generated A-algebra.*

Proposition (6.3.4). — *[Liu, 3.2.4] (i) Every closed immersion is of finite type.*
(ii) The composite of two morphisms of finite type is of finite type.
(iii) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are S-morphisms of finite type, then so is $f \times_S g$.
(iv) If $f\colon X \to Y$ is an S-morphism of finite type, then $f_{(S')}$ is of finite type for any base extension $S' \to S$.
(v) If $g \circ f$ is of finite type, and g is separated, then f is of finite type.
(vi) If f is of finite type, then so is f_{red}.

Corollary (6.3.5). — *[Liu, Ex. 3.2.2] Let $f\colon X \to Y$ be an immersion. If the underlying space of Y is locally Noetherian, or if that of X is Noetherian, then f is of finite type.*

Corollary (6.3.6). — *Given $f: X \to Y$ and $g: Y \to Z$, if $g \circ f$ is of finite type, and if X is Noetherian, or if $X \times_Z Y$ is locally Noetherian, then f is of finite type.*

Proposition (6.3.7). — *If X is of finite type over Y, and Y is (locally) Noetherian, then so is X.*

Corollary (6.3.8). — *If X is of finite type over S, then $X_{(S')}$ is (locally) Noetherian for every base extension $S' \to S$ such that S' is (locally) Noetherian.*

Corollary (6.3.9). — *If X is of finite type over a locally Noetherian prescheme S, then every S-morphism $f: X \to Y$ is of finite type.*

[For morphisms *locally* of finite type, the preceding results hold without the Noetherian hypotheses—see §6.6.]

Proposition (6.3.10). — *A morphism $f: X \to Y$ of finite type is surjective if and only if, for every algebraically closed field k, the map $X(k) \to Y(k)$ induced by f on k-valued points (3.4.1) is surjective.*

[A morphism f satisfying the last condition is said to be *geometrically surjective*.]

6.4. **Algebraic preschemes.**

(6.4.1). Let K be a field. A prescheme X of finite type over K is called an *algebraic* K-prescheme, K the *ground field* of X [Liu, 2.3.47, Example 3.2.3].

An algebraic prescheme is automatically Noetherian.

Proposition (6.4.2). — *Let X be an algebraic K-prescheme. A point $x \in X$ is closed iff $k(x)$ is a finite algebraic extension of K.*

["If" holds for any K-prescheme X and reduces to the fact that an integral domain finite dimensional over K is a field. "Only if" is equivalent to the fact, which is a version of Hilbert's Nullstellensatz, that if $L \supseteq K$ is a field finitely generated as a K algebra, then L is finite algebraic over K.]

Corollary (6.4.3). — *If $K = \overline{K}$ and X is an algebraic K-prescheme, then $X(K)_K \to X$ is a bijection from the set of K-valued points of X to its closed points, which are also its K-rational points.*

Proposition (6.4.4). — *For an algebraic K-prescheme X, the following are equivalent.*

(a) X is Artinian.
(b) The underlying space of X is discrete.
(c) The underlying space of X has finitely many closed points.
(c′) The underlying space of X is finite.
(d) Every point of X is closed.
(e) $X \cong \operatorname{Spec}(A)$ where A is finite-dimensional as a K-vector space.

(6.4.5). When the above hold, we say that X is finite over K, or a finite K-scheme, of length $l_K(X) \underset{def}{=} \dim_K(A)$. If X and Y are finite K-schemes, then

$$l_K(X \coprod Y) = l_K(X) + l_K(Y), \tag{6.4.5.1}$$

$$l_K(X \times_K Y) = l_K(X) l_K(Y). \tag{6.4.5.2}$$

Corollary (6.4.6). — *If X is a finite K-scheme and K' is a finite extension of K, then $X \otimes_K K'$ is finite over K', of length equal to $l_K(X)$.*

Corollary (6.4.7). — *Let X be a finite K-scheme and set $n = \sum_{x \in X} [k(x) : K]_s$. Then for every algebraically closed extension K' of K, the underlying space of $X \otimes_K K'$ has n points, identified bijectively with the set $X(K')_K$ of K'-valued points of X.*

Here $[K : L]_s$ denotes the separable degree of the finite extension $L \subseteq K$, that is, the degree $[K' : L]$, where K' is the maximal separable algebraic extension of L inside K.

(6.4.8). The number n in (6.4.7) is the *separable degree* or the *geometric number of points* of X over K. We have

$$n(X \coprod Y) = n(X) + n(Y), \tag{6.4.8.1}$$

$$n(X \times_K Y) = n(X) n(Y). \tag{6.4.8.2}$$

Proposition (6.4.9). — *Let $f \colon X \to Y$ be a K-morphism of algebraic K-preschemes. Let K' be an algebraically closed extension of infinite transcendence degree over K. Then f is surjective iff $X(K')_K \to Y(K')_K$ is surjective.*

The proof goes by showing that in (6.3.10) it suffices to take k a finitely generated extension of K, hence isomorphic to a subfield of K'.

(6.4.10). In Volume IV it will be shown that the infinite transcendence degree hypothesis is not needed.

Proposition (6.4.11). — *If $f \colon X \to Y$ is of finite type, then for every $y \in Y$, the fiber $f^{-1}(y)$ is algebraic over $k(y)$, and for all closed points $x \in f^{-1}(y)$, $k(x)$ is a finite extension of $k(y)$.*

Proposition (6.4.12). — *Given morphisms $f \colon X \to Y$ and $g \colon Y' \to Y$, let $X' = X \times_Y Y'$ and $f' = f_{(Y')} \colon X' \to Y'$. Let $y' \in Y'$, $y = g(y')$. If the fiber $f^{-1}(y)$ is finite over $k(y)$, then so is $f'^{-1}(y')$ over $k(y')$, with the same degree and geometric number of points as $f^{-1}(y)$.*

(6.4.13). One may understand (6.4.11) as giving the concept of morphism of finite type $f \colon X \to Y$ a geometric significance: it describes a family of algebraic varieties parametrized by points of the target scheme Y.

6.5. **Local determination of a morphism.**

Proposition (6.5.1). — *Let X, Y be S-preschemes, with Y of finite type over S. Suppose $x \in X$, $y \in Y$ lie over the same point $s \in S$.*

(i) If $f, f' \colon X \to Y$ satisfy $f(x) = f'(x) = y$, and they induce the same (local) homomorphism of $\mathcal{O}_{S,s}$-algebras $f_x^\sharp = f_x'^\sharp$ from $\mathcal{O}_{Y,y}$ to $\mathcal{O}_{X,x}$, then f and f' coincide on a neighborhood of x.

(ii) [Liu, Ex. 3.2.4] Suppose further that S is locally Noetherian. Then every local $\mathcal{O}_{S,s}$ algebra homomorphism $\phi \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is induced by an S-morphism f such that $f(x) = y$ from a neighborhood U of x to Y.

Corollary (6.5.2). — *In (6.5.1, (ii)), if X is of finite type over S, one can take f to be of finite type.*

Corollary (6.5.3). — *In (6.5.1, (ii)), if Y is integral and ϕ is injective, one can take $U \cong \operatorname{Spec}(B)$ affine, with $f(U)$ contained in an open affine $W \cong \operatorname{Spec}(A) \subseteq Y$, such that f corresponds to an injective ring homomorphism $\gamma \colon A \to B$.*

Proposition (6.5.4). — *Let $f \colon X \to Y$ be a morphism of finite type, $x \in X$, $y = f(x)$.*

(i) f is a local immersion at x (4.5.1) iff $f_x^\sharp \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective.

(ii) Suppose further that Y is locally Noetherian. Then f is a local isomorphism at x iff $f_x^\sharp$ is an isomorphism.

Corollary (6.5.5). — *Let $f \colon X \to Y$ be of finite type, X irreducible, x its generic point, and $y = f(x)$.*

(i) f is a local immersion at some point of X iff $f_x^\sharp \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective.

(ii) Suppose further that Y is irreducible and locally Noetherian. Then f is a local isomorphism at some point of X iff y is the generic point of Y (which by (0, 2.1.4) means f is dominant), and $f_x^\sharp$ is an isomorphism (that is, f is birational (2.2.9)).

6.6. **Quasi-compact morphisms and morphisms locally of finite type.**

Definition (6.6.1). — [Liu, Ex. 2.3.17] A morphism $f \colon X \to Y$ is *quasi-compact* if $f^{-1}(V)$ is quasi-compact for every quasi-compact open $V \subseteq Y$.

Suppose $\mathfrak{B}$ is a base of the topology on Y which consists of quasi-compact open sets (affines, for example). For f to be quasi-compact, it suffices that $f^{-1}(V)$ be quasi-compact (equivalently, a finite union of affines) for all $V \in \mathfrak{B}$. In particular, if X is quasi-compact and Y is affine, then every morphism $f \colon X \to Y$ is quasi-compact, since for any open affines $V \subseteq Y$ and $U \subseteq X$, $f^{-1}(U) \cap V$ is affine by (5.5.10).

If f is quasi-compact, then so is its restriction $f^{-1}(V) \to V$ for any open $V \subseteq Y$. Conversely, if $Y = \bigcup_\alpha U_\alpha$ is an open covering and each restriction $f^{-1}(U_\alpha) \to U_\alpha$ is quasi-compact, then so is f. In other words, the property that f is quasi-compact is local on Y.

Definition (6.6.2). — A morphism $f \colon X \to Y$ is *locally of finite type* if for every $x \in X$ there are open sets $x \in U \subseteq X$ and $f(U) \subseteq V \subseteq Y$ such that $(f|U) \colon U \to V$ is of finite type.

It is immediate from the definition and (6.3.2) that if f is locally of finite type, then so is its restriction $f^{-1}(V) \to V$ for every open $V \subseteq Y$.

Proposition (6.6.3). — *A morphism f is of finite type if and only if it is quasi-compact and locally of finite type.*

Proposition (6.6.4). — *[Liu, Ex. 2.3.17(a,b)] (i) Every closed immersion is quasi-compact. If the underlying space of X is Noetherian, or if that of Y is locally Noetherian, every immersion $X \to Y$ is quasi-compact.*

(ii) The composite of two quasi-compact morphisms is quasi-compact.

(iii) If $f\colon X \to Y$ is a quasi-compact S-morphism, so is $f_{(S')}$, for any base extension $S' \to S$.

(iv) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are quasi-compact S-morphisms, so is $f \times_S g$.

(v) If the composite $g \circ f$ of $f\colon X \to Y$ and $g\colon Y \to Z$ is quasi-compact, and if g is separated or the underlying space of X is locally Noetherian, then f is quasi-compact.

(vi) f is quasi-compact iff f_{red} is.

Proposition (6.6.5). — *Let $f\colon X \to Y$ be quasi-compact. Then f is dominant iff for each generic point y of an irreducible component of Y, $f^{-1}(y)$ contains the generic point of an irreducible component of X.*

Proposition (6.6.6). — *(i) Every local immersion is locally of finite type.*

(ii) The composite of two morphisms locally of finite type is again so.

(iii) If $f\colon X \to Y$ is an S-morphism locally of finite type, so is $f_{(S')}$, for any base extension $S' \to S$.

(iv) If $f\colon X \to X'$ and $g\colon Y \to Y'$ are S-morphisms locally of finite type, so is $f \times_S g$.

(v) If $g \circ f$ is locally of finite type, then so is f.

(vi) If f is locally of finite type, so is f_{red}.

Corollary (6.6.7). — *Let X, Y be S-preschemes locally of finite type. If S is locally Noetherian, then so is $X \times_S Y$.*

Remark (6.6.8). — Proposition (6.3.10) holds if f is only assumed locally of finite type. Similarly, (6.4.2) and (6.4.9) hold if X, Y are only assumed locally of finite type over K.