Synopsis of material from EGA Chapter I, §5

5. Reduced preschemes; the separation axiom

5.1. **Reduced preschemes.**

Proposition (5.1.1). — *Let $\mathcal{B}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras on a prescheme X. There is a unique quasi-coherent sheaf of ideals $\mathcal{N} \subseteq \mathcal{B}$ such that $\mathcal{N}_x$ is the nilradical of $\mathcal{B}_x$ for all $x \in X$. If $X = \operatorname{Spec} A$ is affine, so $\mathcal{B} = \widetilde{B}$ for an A-algebra B, then $\mathcal{N} = \widetilde{\mathfrak{N}}$, where $\mathfrak{N}$ is the nilradical of B.*

The sheaf $\mathcal{N}$ is called the *nilradical* of $\mathcal{B}$. We write $\mathcal{N}_X$ for the nilradical of $\mathcal{B} = \mathcal{O}_X$.

Corollary (5.1.2). — *The closed sub-prescheme of X defined by the ideal sheaf $\mathcal{N}_X$ is the unique sub-prescheme which is reduced* (0, 4.1.4) *and has underlying space is equal to X; it is the smallest close sub-prescheme with underlying space X.*

Definition (5.1.3). — The closed sub-prescheme in (5.1.2) is called the *associated reduced prescheme* of X, and denoted X_{red}.

Thus X is reduced iff $X = X_{\mathrm{red}}$.

Proposition (5.1.4). — $\operatorname{Spec}(A)$ *is reduced (*resp. *integral)* (2.1.7) *iff A is a reduced ring (*resp. *an integral domain).*

(5.1.5). A morphism $f\colon X \to Y$ induces a unique morphism $f_{\mathrm{red}}\colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ such that the diagram

$$\begin{array}{ccc} X_{\mathrm{red}} & \xrightarrow{f_{\mathrm{red}}} & Y_{\mathrm{red}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, where the vertical arrows are the canonical inclusions of closed sub-preschemes. This makes $X \mapsto X_{\mathrm{red}}$ a functor from preschemes to reduced preschemes. If X is reduced, then every morphism $f\colon X \to Y$ factors uniquely as $X \underset{f_{\mathrm{red}}}{\to} Y_{\mathrm{red}} \to Y$.

[This implies that $X \mapsto X_{\mathrm{red}}$ is right adjoint to the inclusion of the category of reduced preschemes in the category of preschemes.]

Proposition (5.1.6). — *If f is surjective (*resp. *universally injective, an immersion, a closed immersion, an open immersion, a local immersion, a local isomorphism), then so is f_{red}. Conversely, if f_{red} is surjective (*resp. *universally injective), then so is f.*

Proposition (5.1.7). — *Let X, Y be S-preschemes. Then $X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}} = X_{\mathrm{red}} \times_S Y_{\mathrm{red}}$, and it is identified canonically with a sub-prescheme of $X \times_S Y$ whose underlying space is $X \times_S Y$.*

Corollary (5.1.8). — $(X \times_S Y)_{\mathrm{red}} = (X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}})_{\mathrm{red}}$.

Note that even if X and Y are reduced, $X \times_S Y$ need not be.

Proposition (5.1.9). — *Let X be a pre-scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ a quasi-coherent sheaf of ideals such that $\mathcal{I}^n = 0$ for some n. Let $X_0 = (X, \mathcal{O}_X/\mathcal{I})$ be the closed sub-prescheme defined by $\mathcal{I}$. Then X is affine if and only if X_0 is.*

The proof is an application of the vanishing of higher cohomology for quasi-coherent sheaves on an affine scheme.

Corollary (5.1.10). — *If $\mathcal{N}_X$ is nilpotent, then X is affine if and only if X_{red} is.*

5.2. **Existence of a sub-prescheme with a given underlying space.**

Proposition (5.2.1). — *For every locally closed subset $Y \subseteq X$ there exists a unique reduced sub-prescheme of X with underlying space Y.*

Proposition (5.2.2). — *Let X be reduced, $f\colon X \to Y$ a morphism, $Z \subseteq Y$ a closed sub-prescheme such that $f(X) \subseteq Z$. Then f factors through the inclusion $Z \to Y$.*

Corollary (5.2.3). — *Let X be a reduced sub-prescheme of Y, and let Z be the reduced closed sub-prescheme of Y with underlying space $\overline{X}$. Then X is an open sub-prescheme of Z.*

Corollary (5.2.4). — *Let $f\colon X \to Y$ be a morphism, and let X' (*resp. *Y') be a closed sub-prescheme of X (*resp. *Y) defined by an ideal sheaf $\mathcal{I}$ (resp. $\mathcal{J}$). If X' is reduced, and $f(X') \subseteq Y'$, then $f^*(\mathcal{J})\mathcal{O}_X \subseteq \mathcal{I}$.*

5.3. **Diagonal; graph of a morphism.**

(5.3.1). Let X be an S-prescheme. The morphism $\Delta_{X|S} = (1_X, 1_X)\colon X \to X \times_S X$ is called the *diagonal morphism.* We also write Δ_X or just Δ when S and/or X are understood from the context. For any two S-morphisms $f, g\colon T \to X$, note that $(f, g) = (f \times_S g) \circ \Delta_X$.

The definition makes sense and everything in (5.3.1) through (5.3.8) holds in any category where the relevant products exist.

Proposition (5.3.2). — *Under the identification $(X \times Y) \times (X \times Y) = (X \times X) \times (Y \times Y)$, we have $\Delta_{X \times Y} = \Delta_X \times \Delta_Y$.*

[The numbering in EGA skips (5.3.3).]

Corollary (5.3.4). — *For any base extension $S' \to S$, we have $\Delta_{X_{(S')}} = (\Delta_X)_{(S')}$.*

Proposition (5.3.5). — *Let S be a T-prescheme, and let X, Y be S-preschemes (hence also T-preschemes). The diagram*

(5.3.5.1)
$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \times_T Y \\ \downarrow & & \downarrow \\ S & \xrightarrow{\Delta_{S|T}} & S \times_T S \end{array}$$

in which all but the bottom arrow are induced by the structure morphisms identifies $X \times_S Y$ with $(X \times_T Y) \times_{(S \times_T S)} S$.

Corollary (5.3.6). — *The canonical morphism $X \times_S Y \to X \times_T Y$ can be identified with $(1_{X\times_T Y}) \times_P \Delta_S$, where $P = S \times_T S$.*

Corollary (5.3.7). — *If $f\colon X \to Y$ is an S-morphism, the diagram*

$$\begin{array}{ccc} X & \longrightarrow & X \times_S Y \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\Delta_Y} & Y \times_S Y \end{array}$$

identifies X with $(X \times_S Y) \times_{(Y\times_S Y)} Y$.

Proposition (5.3.8). — *For $f\colon X \to Y$ to be a monomorphism it is necessary and sufficient that $\Delta_{X|Y}$ is an isomorphism of X with $X \times_Y X$.*

Proposition (5.3.9). — *The diagonal morphism Δ_X is an immersion of X into $X \times_S X$.*

The image of the diagonal morphism, regarded as a sub-prescheme of $X \times_S X$, is called *the diagonal* in $X \times_S X$.

Corollary (5.3.10). — *The top arrow in* (5.3.5.1) *is an immersion, called the* canonical immersion *of $X \times_T Y$ into $X \times_S Y$.*

Corollary (5.3.11). — *Let $f\colon X \to Y$ be an S-morphism. The graph morphism $\Gamma_f = (1_X, f)$ of f* (3.3.14) *is an immersion of X into $X \times_S Y$. Its image, regarded as a sub-prescheme of $X \times_S Y$, is called the graph of f.*

A sub-prescheme Z of $X \times_S Y$ is the graph of a morphism iff the projection p_1 restricts to an isomorphism $g\colon Z \to X$; then Z is the graph of $p_2 \circ g^{-1}$.

In particular, taking $X = S$, any S-section $S \to Y$ is equal to its graph morphism, and we also refer to its graph (a subscheme of Y) as an S-section of Y.

Corollary (5.3.12). — *Keep the notation of* (5.3.11)*, let $g\colon S' \to S$ be a morphism, and let $f' = f_{(S')}$ be the base change of f by g. Then $\Gamma_{f'} = (\Gamma_f)_{(S')}$.*

Corollary (5.3.13). — *Given morphisms $f\colon X \to Y$, $g\colon Y \to Z$, if $g \circ f$ is an immersion (resp. local immersion), then so is f.*

Corollary (5.3.14). — *Let $j\colon X \to Y$, $g\colon X \to Z$ be S-morphisms. If j is an immersion (resp. local immersion) then so is $(j, g)\colon X \to Y \times_S Z$.*

Proposition (5.3.15). — *Given an S-morphism $f\colon X \to Y$, we have a commutative diagram*

(5.3.15.1)
$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times_S X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\times_S f} \\ Y & \xrightarrow{\Delta_Y} & Y \times_S Y. \end{array}$$

Corollary (5.3.16). — *If X is a sub-prescheme of Y, the diagonal $\Delta_X(X)$ is a sub-prescheme of $\Delta_Y(Y)$, with underlying space*

$$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y(Y) \cap p_2^{-1}(X),$$

where p_1, p_2 are the projections $Y \times_S Y \to Y$.

Corollary (5.3.17). — *Let $f_1, f_2 \colon Y \to X$ be S-morphisms, and $y \in Y$ a point such that $f_1(y) = f_2(y) = x$, and the associated homorphisms $k(x) \to k(y)$ are equal. Then, setting $f = (f_1, f_2)$, the point $f(y)$ belongs to the diagonal $\Delta_X(X)$.*

5.4. **Separated morphisms and preschemes.**

Definition (5.4.1). — [Liu, 3.3.2] A morphism $f \colon X \to Y$ is *separated* if the diagonal morphism $\Delta \colon X \to X \times_Y X$ is a closed immersion. A prescheme X separated over Y is called a *Y-scheme.* A prescheme X is *separated* if it is separated over $\mathrm{Spec}(\mathbb{Z})$. A separated prescheme is called a *scheme*.

By (5.3.9), f is separated if the diagonal $\Delta_X(X)$ is a closed subspace of $X \times_Y X$ [Liu, 3.3.5].

Proposition (5.4.2). — *If $S \to T$ is separated, and X, Y are S-preschemes, the canonical immersion $X \times_S Y \to X \times_T Y$* (5.3.10) *is closed.*

Corollary (5.4.3). — *[Liu, Ex. 3.3.10] If Y is an S-scheme (i.e., a separated S-prescheme) and $f \colon X \to Y$ an S-morphism, the graph morphism Γ_f of f is a closed immersion.*

Corollary (5.4.4). — *If $g \circ f$ is a closed immersion, and g is separated, then f is a closed immersion.*

Corollary (5.4.5). — *Given $j \colon X \to Y$, $g \colon X \to Z$, if Z is an S-scheme, and j is a closed immersion, then so is $(j, g) \colon X \to Y \times_S Z$.*

Corollary (5.4.6). — *If X is an S-scheme, then every S-section of X is a closed immersion.*

Corollary (5.4.7). — *[Liu, 3.3.11] Let S be an integral prescheme with generic point s. Let X be an S-scheme. If S-sections f, g satisfy $f(s) = g(s)$, then $f = g$.*

Remark (5.4.8). — If the conclusion of (5.4.3) holds for $f = 1_Y$, or if (5.4.4) holds for $f = \Delta_{Y|S}$, $g = p_1$, which since $g \circ f = 1_Y$ just means that $p_1 \colon Y \times_S Y \to Y$ is separated, or if (5.4.6) holds for the section Δ_Y of $Y \times_S Y \to Y$ [Liu, Ex. 3.3.7], it follows conversely that $Y \to S$ is separated.

5.5. **Criteria for separation.**

Proposition (5.5.1). — *[Liu, 3.3.9]: (i) Every monomorphism of preschemes (in particular, every immersion) is separated.*

(ii) The composite of separated morphisms is separated.

(iii) If f and g are separated S-morphisms, then so is $f \times_S g$.

(iv) If f is a separated S-morphism, then so is every base change $f_{(S')}$.

(v) If $g \circ f$ is separated, then so is f.

(vi) f is separated if and only if f_{red} (5.1.5) *is separated.*

Corollary (5.5.2). — *If $f\colon X \to Y$ is separated, so is its restriction to any subscheme of X.*

Corollary (5.5.3). — *If X and Y are S-preschemes and Y is separated over S, then $X \times_S Y$ is separated over X.*

Proposition (5.5.4). — *Let X be a prescheme whose underlying space is a finite union of closed subsets X_k. Let $f\colon X \to Y$ be a morphism and for each k let Y_k be a closed subset of Y containing $f(X_k)$. Regard the X_k, Y_k as subschemes of X, Y with the unique reduced pre-scheme structure* (5.2.1)*, so $f|X_k$ factors through a morphism $f_k\colon X_k \to Y_k$ for each k* (5.2.2)*. Then f is separated if and only if each f_k is.*

In particular, if the X_k are the irreducible components of X, we can assume that each Y_k is an irreducible component of Y (0, 2.1.5). The proposition then reduces the question of whether a morphism is separated to the case of integral preschemes (2.1.7).

Proposition (5.5.5). — *Let $Y = \bigcup_\alpha U_\alpha$ be an open covering. Then $f\colon X \to Y$ is separated if and only if all its restrictions $f^{-1}(U_\alpha) \to U_\alpha$ are separated.*

This reduces the question of whether f is separated to the case that Y is affine.

Proposition (5.5.6). — *[Liu, 3.3.6] Let Y be an affine scheme, $X = \bigcup_\alpha U_\alpha$ an open affine covering. A morphism $f\colon X \to Y$ is separated if and only if for all α, β (i) $U_\alpha \cap U_\beta$ is affine, and (ii) the images of the restriction maps $\Gamma(U_\alpha, \mathcal{O}_X) \to \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ and $\Gamma(U_\beta, \mathcal{O}_X) \to \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ generate the ring $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$.*

Corollary (5.5.7). — *[Liu, 3.3.4] Every affine scheme is separated.*

Hence the definition of *scheme* (5.4.1) is consistent with the terminology 'affine scheme.'

Corollary (5.5.8). — *[Liu, Ex. 3.3.2] Let Y be an affine scheme. Then a morphism $f\colon X \to Y$ is separated if and only if X is separated (i.e., X is a scheme).*

Corollary (5.5.9). — *[Liu, Ex. 3.3.8] A morphism $f\colon X \to Y$ is separated if and only if for every separated open sub-prescheme $U \subseteq Y$, the sub-prescheme $f^{-1}(U) \subseteq$ is separated. It suffices that this hold for open affines $U \subseteq Y$.*

Proposition (5.5.10). — *Let Y be a scheme, $f\colon X \to Y$ a morphism. For all open affines $U \subseteq X$, $V \subseteq Y$, $U \cap f^{-1}(V)$ is affine.*

Examples (5.5.11). — The projective line over a field K (2.3.2) is separated by (5.5.6), since $K[x, x^{-1}]$ is generated by its subrings $K[x]$ and $K[x^{-1}]$. The gluing of two copies of $\mathbb{A}^1_K = \operatorname{Spec}(K[x])$ along the identity map on the open set $U = D(x)$ is not separated, since in this case, both subrings in (5.5.6) (ii) are equal to $K[x]$, and they do not generate $K[x, x^{-1}]$.

Remark (5.5.12). — Given a property $\mathbf{P}$ of morphisms of preschemes, consider the following assertions:

(i) Every closed immersion satifies $\mathbf{P}$.

(ii) The composite of two morphisms satisfying **P** satisfies **P**.
(iii) If f, g are S-morphisms satisfying **P**, then $f \times_S g$ satisfies **P**.
(iv) If f satisfies **P**, then so does every base extension $f_{(S')}$.
(v) If $g \circ f$ satifies **P**, and g is separated, then f satisfies **P**.
(vi) If f satisfies **P**, then so does f_{red}.

If (i) and (ii) hold, then (iii) and (iv) are equivalent, and (i)–(iii) imply (v) and (vi). [These implications are used in the proof of (5.5.1) and again later, *e.g.*, in (6.3.4), (6.6.4), (6.6.6).]
Consider also:
(i′) Every immersion satisfies **P**.
(v′) If $g \circ f$ satisfies **P**, then so does f.

Then (i′), (ii), and (iii) imply (v′).

(5.5.13). One also finds that (v) and (vi) follow from (i), (iii) and
(ii′) If j is a closed immersion and g satisfies **P**, then $g \circ j$ satisfies **P**.

Likewise, (v′) follows from (i′), (iii) and
(ii″) If j is an immersion and g satisfies **P**, then $g \circ j$ satisfies **P**.