5. Reduced preschemes; the separation axiom

5.1. Reduced preschemes.

Proposition (5.1.1). — Let $B$ be a quasi-coherent sheaf of $O_X$-algebras on a prescheme $X$. There is a unique quasi-coherent sheaf of ideals $\mathcal{N} \subseteq B$ such that $\mathcal{N}_x$ is the nilradical of $B_x$ for all $x \in X$. If $X = \text{Spec } A$ is affine, so $B = \tilde{B}$ for an $A$-algebra $B$, then $\mathcal{N} = \tilde{\mathfrak{N}}$, where $\tilde{\mathfrak{N}}$ is the nilradical of $\tilde{B}$.

The sheaf $\mathcal{N}$ is called the nilradical of $B$. We write $\mathcal{N}_X$ for the nilradical of $B = O_X$.

Corollary (5.1.2). — The closed sub-prescheme of $X$ defined by the ideal sheaf $\mathcal{N}_X$ is the unique sub-prescheme which is reduced (0, 4.1.4) and has underlying space is equal to $X$; it is the smallest close sub-prescheme with underlying space $X$.

Definition (5.1.3). — The closed sub-prescheme in (5.1.2) is called the associated reduced prescheme of $X$, and denoted $X_{\text{red}}$.

Thus $X$ is reduced iff $X = X_{\text{red}}$.

Proposition (5.1.4). — Spec($A$) is reduced (resp. integral) (2.1.7) iff $A$ is a reduced ring (resp. an integral domain).

(5.1.5). A morphism $f: X \to Y$ induces a unique morphism $f_{\text{red}}: X_{\text{red}} \to Y_{\text{red}}$ such that the diagram

$$
\begin{array}{ccc}
X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

commutes, where the vertical arrows are the canonical inclusions of closed sub-preschemes. This makes $\mathcal{X} \mapsto X_{\text{red}}$ a functor from preschemes to reduced preschemes. If $X$ is reduced, then every morphism $f: X \to Y$ factors uniquely as $X \xrightarrow{f_{\text{red}}} Y_{\text{red}} \to Y$.

[This implies that $\mathcal{X} \mapsto X_{\text{red}}$ is right adjoint to the inclusion of the category of reduced preschemes in the category of preschemes.]

Proposition (5.1.6). — If $f$ is surjective (resp. universally injective, an immersion, a closed immersion, an open immersion, a local immersion, a local isomorphism), then so is $f_{\text{red}}$. Conversely, if $f_{\text{red}}$ is surjective (resp. universally injective), then so is $f$.

Proposition (5.1.7). — Let $X, Y$ be $S$-preschemes. Then $X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}} = X_{\text{red}} \times_S Y_{\text{red}}$, and it is identified canonically with a sub-prescheme of $X \times_S Y$ whose underlying space is $X \times_S Y$.

Corollary (5.1.8). — $(X \times_S Y)_{\text{red}} = (X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}})_{\text{red}}$.

Note that even if $X$ and $Y$ are reduced, $X \times_S Y$ need not be.
Proposition (5.1.9). — Let $X$ be a pre-scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ a quasi-coherent sheaf of ideals such that $\mathcal{I}^n = 0$ for some $n$. Let $X_0 = (X, \mathcal{O}_X/\mathcal{I})$ be the closed sub-prescheme defined by $\mathcal{I}$. Then $X$ is affine if and only if $X_0$ is.

The proof is an application of the vanishing of higher cohomology for quasi-coherent sheaves on an affine scheme.

Corollary (5.1.10). — If $\mathcal{N}_X$ is nilpotent, then $X$ is affine if and only if $X_{\text{red}}$ is.

5.2. Existence of a sub-prescheme with a given underlying space.

Proposition (5.2.1). — For every locally closed subset $Y \subseteq X$ there exists a unique reduced sub-prescheme of $X$ with underlying space $Y$.

Proposition (5.2.2). — Let $X$ be reduced, $f: X \to Y$ a morphism, $Z \subseteq Y$ a closed sub-prescheme such that $f(X) \subseteq Z$. Then $f$ factors through the inclusion $Z \to Y$.

Corollary (5.2.3). — Let $X$ be a reduced sub-prescheme of $Y$, and let $Z$ be the reduced closed sub-prescheme of $Y$ with underlying space $\overline{X}$. Then $X$ is an open sub-prescheme of $Z$.

Corollary (5.2.4). — Let $f: X \to Y$ be a morphism, and let $X'$ (resp. $Y'$) be a closed sub-prescheme of $X$ (resp. $Y$) defined by an ideal sheaf $\mathcal{I}$ (resp. $\mathcal{J}$). If $X'$ is reduced, and $f(X') \subseteq Y'$, then $f^*(\mathcal{J})\mathcal{O}_X \subseteq \mathcal{I}$.

5.3. Diagonal; graph of a morphism.

(5.3.1). Let $X$ be an $S$-prescheme. The morphism $\Delta_{X|S} = (1_X, 1_X): X \to X \times_S X$ is called the diagonal morphism. We also write $\Delta_X$ or just $\Delta$ when $S$ and/or $X$ are understood from the context. For any two $S$-morphisms $f, g: T \to X$, note that $(f, g) = (f \times_S g) \circ \Delta_X$.

The definition makes sense and everything in (5.3.1) through (5.3.8) holds in any category where the relevant products exist.

Proposition (5.3.2). — Under the identification $(X \times Y) \times (X \times Y) = (X \times X) \times (Y \times Y)$, we have $\Delta_{X \times Y} = \Delta_X \times \Delta_Y$.

[The numbering in EGA skips (5.3.3).]

Corollary (5.3.4). — For any base extension $S' \to S$, we have $\Delta_{X|S'} = (\Delta_X)_{|S'}$.

Proposition (5.3.5). — Let $S$ be a $T$-prescheme, and let $X, Y$ be $S$-preschemes (hence also $T$-preschemes). The diagram

\[
\begin{array}{ccc}
X \times_S Y & \longrightarrow & X \times_T Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{\Delta_{S|T}} & S \times_T S
\end{array}
\]

(5.3.5.1)

in which all but the bottom arrow are induced by the structure morphisms identifies $X \times_S Y$ with $(X \times_T Y) \times (S \times_T S)$.
Corollary (5.3.6). — The canonical morphism \( X \times_S Y \to X \times_T Y \) can be identified with \((1_{X\times T Y}) \times_P \Delta_S\), where \( P = S \times T S \).

Corollary (5.3.7). — If \( f: X \to Y \) is an \( S \)-morphism, the diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X \times_S Y \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\Delta_Y} & Y \times_S Y \\
\end{array}
\]

identifies \( X \) with \( (X \times_S Y) \times_{(Y \times_S Y)} Y \).

Proposition (5.3.8). — For \( f: X \to Y \) to be a monomorphism it is necessary and sufficient that \( \Delta_{X|Y} \) is an isomorphism of \( X \) with \( X \times_Y X \).

Proposition (5.3.9). — The diagonal morphism \( \Delta_X \) is an immersion of \( X \) into \( X \times_S X \).

The image of the diagonal morphism, regarded as a sub-prescheme of \( X \times_S X \), is called the diagonal in \( X \times_S X \).

Corollary (5.3.10). — The top arrow in (5.3.5.1) is an immersion, called the canonical immersion of \( X \times_T Y \) into \( X \times_S Y \).

Corollary (5.3.11). — Let \( f: X \to Y \) be an \( S \)-morphism. The graph morphism \( \Gamma_f = (1_X, f) \) of \( f \) (3.3.14) is an immersion of \( X \) into \( X \times_S Y \). Its image, regarded as a sub-prescheme of \( X \times_S Y \), is called the graph of \( f \).

A sub-prescheme \( Z \) of \( X \times_S Y \) is the graph of a morphism iff the projection \( p_1 \) restricts to an isomorphism \( g: Z \to X \); then \( Z \) is the graph of \( p_2 \circ g^{-1} \).

In particular, taking \( X = S \), any \( S \)-section \( S \to Y \) is equal to its graph morphism, and we also refer to its graph (a subscheme of \( Y \)) as an \( S \)-section of \( Y \).

Corollary (5.3.12). — Keep the notation of (5.3.11), let \( g: S' \to S \) be a morphism, and let \( f' = f_{(S')} \) be the base change of \( f \) by \( g \). Then \( \Gamma_{f'} = (\Gamma_f)_{(S')} \).

Corollary (5.3.13). — Given morphisms \( f: X \to Y \), \( g: Y \to Z \), if \( g \circ f \) is an immersion (resp. local immersion), then so is \( f \).

Corollary (5.3.14). — Let \( j: X \to Y \), \( g: X \to Z \) be \( S \)-morphisms. If \( j \) is an immersion (resp. local immersion) then so is \( (j, g): X \to Y \times_S Z \).

Proposition (5.3.15). — Given an \( S \)-morphism \( f: X \to Y \), we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_X} & X \times_S X \\
\downarrow f & & \downarrow f \times_S f \\
Y & \xrightarrow{\Delta_Y} & Y \times_S Y \\
\end{array}
\]
Corollary (5.3.16). — If $X$ is a sub-prescheme of $Y$, the diagonal $\Delta_X(X)$ is a sub-prescheme of $\Delta_Y(Y)$, with underlying space
\[ \Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y(Y) \cap p_2^{-1}(X), \]
where $p_1$, $p_2$ are the projections $Y \times_SY \to Y$.

Corollary (5.3.17). — Let $f_1, f_2: Y \to X$ be $S$-morphisms, and $y \in Y$ a point such that $f_1(y) = f_2(y) = x$, and the associated homomorphisms $k(x) \to k(y)$ are equal. Then, setting $f = (f_1, f_2)$, the point $f(y)$ belongs to the diagonal $\Delta_X(X)$.

5.4. Separated morphisms and preschemes.

Definition (5.4.1). — [Liu, 3.3.2] A morphism $f: X \to Y$ is separated if the diagonal morphism $\Delta: X \to X \times_Y X$ is a closed immersion. A prescheme $X$ separated over $Y$ is called a $Y$-scheme. A prescheme $X$ is separated if it is separated over $\text{Spec}(Z)$. A separated prescheme is called a scheme.

By (5.3.9), $f$ is separated if the diagonal $\Delta_X(X)$ is a closed subspace of $X \times_Y X$ [Liu, 3.3.5].

Proposition (5.4.2). — If $S \to T$ is separated, and $X$, $Y$ are $S$-preschemes, the canonical immersion $X \times_SY \to X \times_T Y$ (5.3.10) is closed.

Corollary (5.4.3). — [Liu, Ex. 3.3.10] If $Y$ is an $S$-scheme (i.e., a separated $S$-prescheme) and $f: X \to Y$ an $S$-morphism, the graph morphism $\Gamma_f$ of $f$ is a closed immersion.

Corollary (5.4.4). — If $g \circ f$ is a closed immersion, and $g$ is separated, then $f$ is a closed immersion.

Corollary (5.4.5). — Given $j: X \to Y$, $g: X \to Z$, if $Z$ is an $S$-scheme, and $j$ is a closed immersion, then so is $(j,g): X \to Y \times_SZ$.

Corollary (5.4.6). — If $X$ is an $S$-scheme, then every $S$-section of $X$ is a closed immersion.

Corollary (5.4.7). — [Liu, 3.3.11] Let $S$ be an integral prescheme with generic point $s$. Let $X$ be an $S$-scheme. If $S$-sections $f$, $g$ satisfy $f(s) = g(s)$, then $f = g$.

Remark (5.4.8). — If the conclusion of (5.4.3) holds for $f = 1_Y$, or if (5.4.4) holds for $f = \Delta_{Y|S}$, $g = p_1$, which since $g \circ f = 1_Y$ just means that $p_1: Y \times_SY \to Y$ is separated, or if (5.4.6) holds for the section $\Delta_Y$ of $Y \times_SY \to Y$ [Liu, Ex. 3.3.7], it follows conversely that $Y \to S$ is separated.

5.5. Criteria for separation.

Proposition (5.5.1). — [Liu, 3.3.9]: (i) Every monomorphism of preschemes (in particular, every immersion) is separated.

(ii) The composite of separated morphisms is separated.

(iii) If $f$ and $g$ are separated $S$-morphisms, then so is $f \times_S g$.

(iv) If $f$ is a separated $S$-morphism, then so is every base change $f_{(S')}$. 
(v) If \( g \circ f \) is separated, then so is \( f \).
(vi) \( f \) is separated if and only if \( f_{\text{red}} \) \((5.1.5)\) is separated.

**Corollary (5.5.2).** — If \( f : X \to Y \) is separated, so is its restriction to any subscheme of \( X \).

**Corollary (5.5.3).** — If \( X \) and \( Y \) are \( S \)-preschemes and \( Y \) is separated over \( S \), then \( X \times_S Y \) is separated over \( X \).

**Proposition (5.5.4).** — Let \( X \) be a prescheme whose underlying space is a finite union of closed subsets \( X_k \). Let \( f : X \to Y \) be a morphism and for each \( k \) let \( Y_k \) be a closed subset of \( Y \) containing \( f(X_k) \). Regard the \( X_k, Y_k \) as subschemes of \( X, Y \) with the unique reduced pre-scheme structure \((5.2.1)\), so \( f|X_k \) factors through a morphism \( f_k \) : \( X_k \to Y_k \) for each \( k \) \((5.2.2)\). Then \( f \) is separated if and only if each \( f_k \) is.

In particular, if the \( X_k \) are the irreducible components of \( X \), we can assume that each \( Y_k \) is an irreducible component of \( Y \) \((0, 2.1.5)\). The proposition then reduces the question of whether a morphism is separated to the case of integral preschemes \((2.1.7)\).

**Proposition (5.5.5).** — Let \( Y = \bigcup_\alpha U_\alpha \) be an open covering. Then \( f : X \to Y \) is separated if and only if all its restrictions \( f^{-1}(U_\alpha) \to U_\alpha \) are separated.

This reduces the question of whether \( f \) is separated to the case that \( Y \) is affine.

**Proposition (5.5.6).** — [Liu, 3.3.6] Let \( Y \) be an affine scheme, \( X = \bigcup_\alpha U_\alpha \) an open affine covering. A morphism \( f : X \to Y \) is separated if and only if for all \( \alpha, \beta \) \((i) U_\alpha \cap U_\beta \) is affine, and \((ii)\) the images of the restriction maps \( \Gamma(U_\alpha, \mathcal{O}_X) \to \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X) \) and \( \Gamma(U_\beta, \mathcal{O}_X) \to \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X) \) generate the ring \( \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X) \).

**Corollary (5.5.7).** — [Liu, 3.3.4] Every affine scheme is separated.

Hence the definition of scheme \((5.4.1)\) is consistent with the terminology ‘affine scheme.’

**Corollary (5.5.8).** — [Liu, Ex. 3.3.2] Let \( Y \) be an affine scheme. Then a morphism \( f : X \to Y \) is separated if and only if \( X \) is separated (i.e., \( X \) is a scheme).

**Corollary (5.5.9).** — [Liu, Ex. 3.3.8] A morphism \( f : X \to Y \) is separated if and only if for every separated open sub-prescheme \( U \subseteq Y \), the sub-prescheme \( f^{-1}(U) \subseteq X \) is separated.

It suffices that this hold for open affines \( U \subseteq Y \).

**Proposition (5.5.10).** — Let \( Y \) be a scheme, \( f : X \to Y \) a morphism. For all open affines \( U \subseteq X, V \subseteq Y \), \( U \cap f^{-1}(V) \) is affine.

**Examples (5.5.11).** — The projective line over a field \( K \) \((2.3.2)\) is separated by \((5.5.6)\), since \( K[x, x^{-1}] \) is generated by its subrings \( K[x] \) and \( K[x^{-1}] \). The gluing of two copies of \( \mathbb{A}^1_K = \text{Spec}(K[x]) \) along the identity map on the open set \( U = D(x) \) is not separated, since in this case, both subrings in \((5.5.6)\) \((ii)\) are equal to \( K[x] \), and they do not generate \( K[x, x^{-1}] \).

**Remark (5.5.12).** — Given a property \( P \) of morphisms of preschemes, consider the following assertions:

(i) Every closed immersion satisfies \( P \).
(ii) The composite of two morphisms satisfying $P$ satisfies $P$.
(iii) If $f$, $g$ are $S$-morphisms satisfying $P$, then $f \times_S g$ satisfies $P$.
(iv) If $f$ satisfies $P$, then so does every base extension $f_{(S')}$.
(v) If $g \circ f$ satisfies $P$, and $g$ is separated, then $f$ satisfies $P$.
(vi) If $f$ satisfies $P$, then so does $f_{\text{red}}$.

If (i) and (ii) hold, then (iii) and (iv) are equivalent, and (i)–(iii) imply (v) and (vi). [These implications are used in the proof of (5.5.1) and again later, e.g., in (6.3.4), (6.6.4), (6.6.6).]

Consider also:

(i') Every immersion satisfies $P$.
(v') If $g \circ f$ satisfies $P$, then so does $f$.

Then (i'), (ii), and (iii) imply (v').

(5.5.13). One also finds that (v) and (vi) follow from (i), (iii) and
(ii') If $j$ is a closed immersion and $g$ satisfies $P$, then $g \circ j$ satisfies $P$.
Likewise, (v') follows from (i'), (iii) and
(ii'') If $j$ is an immersion and $g$ satisfies $P$, then $g \circ j$ satisfies $P$. 