Synopsis of material from EGA Chapter I, §4

4. Sub-preschemes and immersion morphisms

4.1. **Sub-preschemes.**

(4.1.1). On a prescheme X, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent iff for all open affines $U \subseteq X$, $\mathcal{F}|U$ is isomorphic to the sheaf associated to a $\Gamma(U, \mathcal{O}_X)$-module. In particular, $\mathcal{O}_X$ itself is quasi-coherent, and quasi-coherence is preserved by kernels, cokernels and images of homomorphisms, as well as inductive limits and direct sums (1.3.7, 1.3.9).

Proposition (4.1.2). — *Let X be a prescheme, $\mathcal{I} \subseteq \mathcal{O}_X$ a quasi-coherent sheaf of ideals. The support Y of $\mathcal{O}_X/\mathcal{I}$ is closed, and if $\mathcal{O}_Y$ denotes the restriction of $\mathcal{O}_X/\mathcal{I}$ to Y, then $(Y, \mathcal{O}_Y)$ is a prescheme.*

We say that $(Y, \mathcal{O}_Y)$ as above is the *sub-prescheme of $(X, \mathcal{O}_X)$ defined by the sheaf of ideals $\mathcal{I}$.*

Definition (4.1.3). — $(Y, \mathcal{O}_Y)$ is a *sub-prescheme* of a prescheme $(X, \mathcal{O}_X)$ if:

(1) Y is a locally closed subspace of X;

(2) If U is the largest open set containing Y and in which Y is closed, then $(Y, \mathcal{O}_Y)$ is the sub-prescheme of $(U, \mathcal{O}_X|U)$ defined by a quasi-coherent sheaf of ideals in $\mathcal{O}_X|U$.

If Y is closed (*i.e.*, $U = X$), Y is a *closed sub-prescheme.* Closed sub-preschemes are in one-to-one correspondence with quasi-coherent sheaves of ideals. If $Y = (U, \mathcal{O}_X|U)$, Y is an *open sub-prescheme.*

(4.1.4). With Y, U as above, if $V \subseteq U$ is an open subset, then $(Y \cap V, \mathcal{O}_Y|(Y \cap V))$ is again a subscheme of X. Conversely:

Proposition (4.1.5). — *Let $(Y, \mathcal{O}_Y)$ be a ringed space with Y a subspace of X. If there is a covering of Y by open subsets V_α of X such that each $(Y \cap V_\alpha, \mathcal{O}_Y|(Y \cap V_\alpha))$ is a closed sub-prescheme of V_α, then Y is a sub-prescheme of X.*

Proposition (4.1.6). — *A sub-prescheme of a sub-prescheme is (canonically identified with) a sub-prescheme. Likewise for closed sub-preschemes.*

(4.1.7). Let Y be a sub-prescheme of X, $j\colon Y \hookrightarrow X$ the inclusion of Y as a subspace, so $j^{-1}(\mathcal{O}_X)$ is the restriction of $\mathcal{O}_X$ to Y. We have a canonical surjective sheaf homomorphism $\phi^\sharp\colon j^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$, and thus a monomorphism $(j, \phi)\colon Y \hookrightarrow X$ of preschemes, called the *canonical injection.*

Given any morphism $f\colon X \to Z$, we call $f \circ j\colon Y \to Z$ the *restriction* of f to Y.

(4.1.8). A morphism $f\colon Z \to X$ is said to be *majorized* by the inclusion $j\colon Y \hookrightarrow X$ of a sub-prescheme if f factors as $Z \underset{g}{\to} Y \underset{j}{\to} X$. Here g is unique, because j is a monomorphism.

Proposition (4.1.9). — *For $f\colon Z \to X$ to be majorized by $j\colon Y \to X$ it is necessary and sufficient that $f(Z) \subseteq Y$, and that for every $z \in Z$, setting $y = f(z)$, the kernel of the homomorphism $\mathcal{O}_{X,y} \to \mathcal{O}_{Z,z}$ given by f contains the kernel of $\mathcal{O}_{X,y} \to \mathcal{O}_{Y,y}$.*

Corollary (4.1.10). — *The inclusion of a sub-prescheme $Z \hookrightarrow X$ is majorized by the inclusion of another sub-prescheme $Y \hookrightarrow X$ if and only if Z is a sub-prescheme of Y.*

In this case we write $Z \leq Y$.

4.2. **Immersion morphisms.**

Definition (4.2.1). — An *immersion* is a morphism $f\colon Y \to X$ which factors as $Y \underset{g}{\to} Z \underset{j}{\to} X$, where g is an isomorphism, and $j\colon Z \to X$ is the inclusion of a sub-prescheme. An immersion is *closed* (resp. *open*) if Z is a closed (resp. open) sub-prescheme.

The factorization is unique, and an immersion is a monomorphism, hence universally injective (3.5.4).

Proposition (4.2.2). — *(a) $(f, \phi)\colon Y \to X$ is an open immersion iff f is a homeomorphism onto an open subset of X, and for every $y \in Y$, $\phi_y^\sharp\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is an isomorphism.*

(b) $(f, \phi)\colon Y \to X$ is an immersion (resp. closed immersion) iff f is a homeomorphism onto a locally closed (resp. closed) subset of X, and for every $y \in Y$, $\phi_y^\sharp\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is surjective.

Corollary (4.2.3). — *Let X be an affine scheme. Then $f\colon Y \to X$ is a closed immersion iff Y is affine and $\Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_Y)$ is surjective.*

Corollary (4.2.4). — *(a) Let $f\colon Y \to X$ be a morphism and let (V_α) be a covering of $f(Y)$ by open subsets of X. Then f is an (open) immersion iff every $f^{-1}(V_\alpha) \to V_\alpha$ is an (open) immersion.*

(b) Assume that (V_α) is a covering of X. Then f is a closed immersion iff every $f^{-1}(V_\alpha) \to V_\alpha$ is a closed immersion.

Proposition (4.2.5). — *The composite of two immersions is an immersion; likewise for open or closed immersions.*

4.3. **Products of immersions.**

Proposition (4.3.1). — *If S-morphisms $\alpha\colon X' \to X$, $\beta\colon Y' \to Y$ are immersions, then so is $\alpha \times_S \beta$. Moreover, identifying X', Y' with sub-preschemes of X, Y, the underlying space of $X' \times_S Y'$ is identified with the subspace $p^{-1}(X') \cap q^{-1}(Y') \subseteq X \times_S Y$, where p, q are the projections. Likewise for open or closed immersions.*

Corollary (4.3.2). — *If $f\colon X \to Y$ is an immersion, then so is any base change $f_{(S')}$. Likewise for open or closed immersions.*

4.4. **Preimage of a sub-prescheme.**

Proposition (4.4.1). — *Let $f\colon X \to Y$ be a morphism, $j\colon Y' \to Y$ the inclusion of a sub-prescheme. Then the projection $p\colon X \times_Y Y' \to X$ is an immersion, whose image is a sub-prescheme with underlying space $f^{-1}(Y')$. Moreover, letting j' denote the inclusion of this sub-prescheme, a morphism $h\colon Z \to X$ is majorized by j' if and only if $f \circ h\colon Z \to Y$ is majorized by j. Likewise for open or closed sub-preschemes and immersions.*

From now on, we regard $f^{-1}(Y')$ as endowed with the structure of sub-prescheme of X provided by the proposition.

We have $f^{-1}(Y') = X$ (as a sub-prescheme) if and only if f is majorized by $j\colon Y' \to Y$.

If $y \in Y$ is a closed point, and $Y' = \operatorname{Spec} k(y)$, then $f^{-1}(Y')$ is the fiber $f^{-1}(y)$, with the prescheme structure defined in (3.6.2).

Corollary (4.4.2). — *Given $f\colon X \to Y$, $g\colon Y \to Z$, set $h = g \circ f$. For any sub-prescheme $Z' \subseteq Z$, we have $f^{-1}(g^{-1}(Z')) = h^{-1}(Z')$ as sub-preschemes of X.*

Corollary (4.4.3). — *Let $j'\colon X' \to X$, $j''\colon X'' \to X$ be inclusions of sub-preschemes. Then $j'^{-1}(X'') \cong j''^{-1}(X') \cong X' \times_X X''$ is the greatest lower bound* $\inf(X', X'')$ *in the ordering $\leq$ on sub-preschemes of X.*

[These days it is usual to write $X' \cap X''$ for $\inf(X', X'')$, and refer to it as the *scheme-theoretic intersection* of the two subschemes.]

Corollary (4.4.4). — *Given $f\colon X \to Y$ and two sub-preschemes $Y', Y'' \subseteq Y$, we have $f^{-1}(\inf(Y', Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y''))$.*

Corollary (4.4.5). — *Given $f\colon X \to Y$ and a closed sub-prescheme $Y' \subseteq Y$ defined by a quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$, the sub-prescheme $f^{-1}(Y')$ is defined by the ideal sheaf $f^{-1}(\mathcal{I})\mathcal{O}_X \subseteq \mathcal{O}_X$.*

Corollary (4.4.6). — *Let $j\colon X' \to X$ be a closed sub-prescheme defined by a quasi-coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$. Then the restriction of $f\colon X \to Y$ to X' is majorized by $j'\colon Y' \to Y$ if and only if $f^{-1}(\mathcal{I})\mathcal{O}_X \subseteq \mathcal{J}$.*

4.5. **Local immersions and local isomorphisms.**

Definition (4.5.1). — $f\colon X \to Y$ is a *local immersion at $x \in X$* if there are open neighborhoods U of x and V of $f(x)$ such that the restriction of f to U is a closed immersion $U \hookrightarrow V$. We say that f is a *local immersion* if it is a local immersion at every point.

Definition (4.5.2). — $f\colon X \to Y$ is a *local isomorphism at $x \in X$* if there is an open neighborhood U of x such that the restriction of f to U is an open immersion into Y. We say that f is a *local isomorphism* if it is a local isomorphism at every point.

(4.5.3). A (closed) immersion can then be characterized as a local immersion $f\colon X \to Y$ such that f is a homeomorphism of X onto a (closed) subspace of Y; an open immersion can be characterized as an injective local isomorphism.

Proposition (4.5.4). — *Let X be irreducible, and $f\colon X \to Y$ a dominant, injective morphism. If f is a local immersion, then f is an immersion and $f(X)$ is open in Y.*

Proposition (4.5.5). — *(i) A composite of local immersions (resp. local isomorphisms) is a local immersion (resp. local isomorphism).*

(ii) If S-morphisms f, g are local immersions (resp. local isomorphisms), then so is $f \times_S g$.

(iii) Any base change $f_{(S')}$ of a local immersion (resp. local isomorphism) is a local immersion (resp. local isomorphism).