Synopsis of material from EGA Chapter I, §2

2. Preschemes and morphisms of preschemes

[Note on terminology: today the term *scheme* is usually used for what EGA calls a *prescheme*. What EGA calls a scheme is now called a *separated scheme*. Liu, in particular, uses the current terminology.]

2.1. **Definition of preschemes.**

(2.1.1). An open subset V of a ringed space X is called *affine open* if $(V, \mathcal{O}_X|V)$ is an affine scheme (1.7.1).

Definition (2.1.2). — [Liu, 2.3.8] A *prescheme* is a ringed space $(X, \mathcal{O}_X)$ such that every point has an open affine neighborhood.

Proposition (2.1.3). — *The open affine subsets of a prescheme form a base of its topology.*

Proposition (2.1.4). — *The underlying space of a prescheme is T_0.*

Proposition (2.1.5). — *Every irreducible closed subset of a prescheme X has a unique generic point; thus $x \to \overline{\{x\}}$ is a bijection from X to its set of irreducible closed subsets [Liu, 2.4.12 is a special case].*

(2.1.6). If y is the generic point of an irreducible closed subset $Y \subseteq X$, we sometimes write $\mathcal{O}_{X/Y}$ for $\mathcal{O}_{X,y}$ and call it the local ring of X along Y, or the local ring of Y in X.
If X is itself irreducible, with generic point x, then $\mathcal{O}_{X,x}$ is called the *ring of rational functions* on X.

Proposition (2.1.7). — *[Liu, 2.3.9] If X is a prescheme and $U \subseteq X$ is open, then $(U, \mathcal{O}_X|U)$ is a prescheme.*
This follows from (2.1.3).

(2.1.8). A prescheme X is *irreducible*, or *connected*, if its underlying space is. X is *integral* if it is irreducible and reduced (cf. (5.1.4)) [Liu, 2.4.16]. X is *locally integral* if every $x \in X$ has an open neighborhood which is integral.

2.2. **Morphisms of preschemes.**

Definition (2.2.1). — [Liu, 2.3.13] A *morphism of preschemes* is a morphism of ringed spaces $(f, \phi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $\phi_x^\sharp\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings for all $x \in X$.
In particular, $\phi_x^\sharp$ induces a homomorphism $\phi^x\colon k(f(x)) \to k(x)$, making the field $k(x)$ an extension of $k(f(x))$.
[In other words, a morphism of preschemes $f\colon X \to Y$ is by definition a local morphism between the locally ringed spaces X, Y.]

(2.2.2). Morphism are closed under composition, making preschemes into a category.

Example (2.2.3). — If $U \subseteq X$ is open, the inclusion of $(U, \mathcal{O}_X|U)$ as an open sub-prescheme of X is a morphism from U to X. By (0, 4.1.1) this is a monomorphism in the category of ringed spaces and hence also in the category of preschemes.

Proposition (2.2.4). — *[Liu, 2.3.25] Let $(X, \mathcal{O}_X)$ be a prescheme and $(S, \mathcal{O}_S) = \mathrm{Spec}(A)$ an affine scheme. Then there is a canonical bijection between morphisms $X \to S$ and ring homomorphisms $A \to \mathcal{O}_X(X)$.*

[This holds more generally for any locally ringed space $(X, \mathcal{O}_X)$. See §1.8.]

Proposition (2.2.5). — *Let $f\colon X \to S$ correspond to $\phi\colon A \to \mathcal{O}_X(X)$ as in (2.2.4). Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a quasi-coherent sheaf of $\mathcal{O}_X$-modules (resp. $\mathcal{O}_Y$-modules), and let $M = \mathcal{F}(S)$ [so $\mathcal{F} = \widetilde{M}$]. Then f-morphisms $\mathcal{F} \to \mathcal{G}$ (0, 4.4.1) are in natural bijection with A-module homomorphisms $M \to \mathcal{G}(X)$.*

(2.2.6). A morphism $(f, \phi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is said to be *open* if $f(U)$ is open for every open $U \subseteq X$, *closed* if $f(Z)$ is closed for every closed $Z \subseteq X$, *dominant* if $f(X)$ is dense in Y, *surjective* if f is surjective. These conditions are properties of f alone.

Proposition (2.2.7). — *Let $X \underset{f}{\to} Y \underset{g}{\to} Z$ be morphisms of presechemes.*

(i) If f and g are open (resp. closed, dominant, surjective), then so is $g \circ f$.

(ii) If f is surjective and $g \circ f$ is closed, then g is closed.

(iii) If $g \circ f$ is surjective, then g is surjective.

Proposition (2.2.8). — *Given a morphism $f\colon X \to Y$ and an open covering $Y = \bigcup_\alpha U_\alpha$, let $f_\alpha\colon f^{-1}(U_\alpha) \to U_\alpha$ be the restriction of f. Then f is open (resp. closed, dominant, surjective) if and only if every f_α satisfies the same condition.*

[In other words, the conditions that f is open, etc., are *local on* Y.]

(2.2.9). Suppose X and Y have the same, finite, number of irreducible components X_i, Y_i, $1 \le i \le n$. Let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i). A morphism $(f, \phi) : X \to Y$ is called *birational* if $f^{-1}(\{\eta_i\}) = \{\xi_i\}$ and $\phi^\sharp_{\xi_i} : \mathcal{O}_{\eta_i} \to \mathcal{O}_{\xi_i}$ is an isomorphism, for each i.

A birational morphism is dominant, hence surjective if it is closed.

(2.2.10). We often write just f for a morphism (f, ϕ) and U for an open subscheme $(U, \mathcal{O}_X|U)$.

2.3. **Gluing preschemes.**

(2.3.1). [Liu, 2.3.33] A ringed space constructed by gluing preschemes (0, 4.1.7) is again a prescheme. Every prescheme is a gluing of affine schemes.

Example (2.3.2). — [Liu, 2.3.34] Let K be a field, $B = K[s]$, $C = K[t]$, $X_1 = \mathrm{Spec}(B)$, $X_2 = \mathrm{Spec}(C)$. Let $U_{12} = D(s) \subset X_1$, $U_{21} = D(t) \subset X_2$, so $U_{12} = \mathrm{Spec}(K[s, s^{-1}])$, $U_{21} = \mathrm{Spec}(K[t, t^{-1}])$. Let $u_{12}\colon U_{21} \to U_{12}$ correspond to the isomorphism $K[t, t^{-1}] \to K[s, s^{-1}]$ given by $t \mapsto 1/s$. Gluing X_1 and X_2 along u_{12} gives the *projective line* $X = \mathbb{P}^1(K)$, a special case of a more general construction (II, 2.4.3). One proves that $\Gamma(X, \mathcal{O}_X) = K$, hence X is not an affine scheme, as it would then be reduced to a point.

2.4. **Local schemes.**

(2.4.1). A *local scheme* is an affine scheme $X = \operatorname{Spec}(A)$ where A is a local ring. Then X has a unique closed point a, and $a \in \overline{\{b\}}$ for all $b \in X$.

For any point y of a prescheme Y, $\operatorname{Spec}(\mathcal{O}_y)$ is called the *local scheme of* Y *at* y. For any affine neighborhood $V = \operatorname{Spec}(B)$ of y, we have $\mathcal{O}_y \cong B_y$, and $B \to B_y$ induces a morphism $\operatorname{Spec}(\mathcal{O}_y) \to V$. Composing this with the inclusion $V \hookrightarrow Y$ gives a canonical morphism $\operatorname{Spec}(\mathcal{O}_y) \to Y$ independent of the choice of V [Liu, 2.3.16].

Proposition (2.4.2). — *Let* $(f, \phi)\colon (\operatorname{Spec}(\mathcal{O}_y), \widetilde{\mathcal{O}_y}) \to (Y, \mathcal{O}_Y)$ *be the canonical morphism. Then* f *is a homeomorphism of* $\operatorname{Spec}(\mathcal{O}_y)$ *onto the subspace* $S_y \subseteq Y$ *consisting of points* z *such that* $y \in \overline{\{z\}}$*, and if* $z = f(\mathfrak{p})$*, then* $\phi_z^\sharp\colon \mathcal{O}_z \to (\mathcal{O}_y)_\mathfrak{p}$ *is an isomorphism. Hence* (f, ϕ) *is a monomorphism of ringed spaces.*

In particular, $\operatorname{Spec}(\mathcal{O}_y)$ *is in bijection with the set of irreducible closed subsets of* Y *that contain* y*.*

Corollary (2.4.3). — *A point* $y \in Y$ *is the generic point of an irreducible component of* Y *if and only if the maximal ideal of* $\mathcal{O}_y$ *is its unique prime ideal (in other words,* $\mathcal{O}_y$ *has Krull dimension zero).*

Proposition (2.4.4). — *Let* $X = \operatorname{Spec}(A)$ *be a local scheme,* a *its unique closed point,* Y *a prescheme. Every morphism* $f\colon X \to Y$ *factors uniquely as* $X \to \operatorname{Spec}(\mathcal{O}_{f(a)}) \to Y$*. This gives a bijective correspondence between morphisms* $X \to Y$*, and pairs consisting of a point* $y \in Y$ *and a local homomorphism of local rings* $\mathcal{O}_y \to A$*.*

(2.4.5). If K is a field, $\operatorname{Spec}(K)$ has only one point. If A is local with maximal ideal $\mathfrak{m}$, then every local homomorphism $A \to K$ factors as $A \to A/\mathfrak{m} \to K$. Hence morphisms $\operatorname{Spec}(A) \to \operatorname{Spec}(K)$ are in bijection with homomorphisms of fields $A/\mathfrak{m} \to K$.

Given a prescheme Y, a point $y \in Y$, and an ideal $\mathfrak{a}_y \subseteq \mathcal{O}_y$, composing the canonical morphisms $\operatorname{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to \operatorname{Spec}(\mathcal{O}_y) \to Y$ gives a canonical morphism $\operatorname{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to Y$. In particular, for $\mathfrak{a}_y = \mathfrak{m}_y$, we get $\operatorname{Spec}(k(y)) \to Y$.

Corollary (2.4.6). — *Let* $X = \operatorname{Spec}(K) = \{\xi\}$*, where* K *is a field. Every morphism* $u\colon X \to Y$ *factors uniquely as* $X \to \operatorname{Spec}(k(u(\xi))) \to Y$*. This gives a bijective correspondence between morphisms* $X \to Y$*, and pairs consisting of a point* $y \in Y$ *and a field extension* $k(y) \hookrightarrow K$*.*

Corollary (2.4.7). — *The canonical morphism* $\operatorname{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to Y$ *is a monomorphism of ringed spaces.*

Remark (2.4.8). — Let X be a local scheme, with closed point a. The only affine open subset of X containing a is X itself. Hence an invertible sheaf (0, 5.4.1) of $\mathcal{O}_X$-modules is necessarily trivial, *i.e.*, isomorphic to $\mathcal{O}_X$. This property does not hold for a general affine scheme $\operatorname{Spec}(A)$. If A is a *normal domain*, it is equivalent to A having *unique factorization.*

2.5. **Preschemes over a prescheme.**

Definition (2.5.1). — [Liu, 2.3.21] Fix a prescheme S. A *prescheme over* S, or S-*prescheme*, is a prescheme X together with a morphism $\phi\colon X \to S$. One says that S is the *base prescheme*, and ϕ is the *structure morphism*. If $S = \operatorname{Spec}(A)$ one also calls X a *prescheme over* A, or an A-*prescheme*.

By (2.2.4), to give an A-prescheme it is equivalent to give a prescheme X whose structure sheaf $\mathcal{O}_X$ is a sheaf of A-algebras [Liu, 2.3.26]. In particular, *every prescheme is a* $\mathbb{Z}$-*prescheme in a unique way* [Liu, 2.3.27].

If $\phi(x) = s$, we say that the point $x \in X$ *lies over* $s \in S$. If ϕ is dominant (2.2.6), we say that X *dominates* S.

(2.5.2). Given S-preschemes X and Y, a morphism $u\colon X \to Y$ is a *morphism of* S-*preschemes*, or S-*morphism*, if $\phi' \circ u = \phi$, where ϕ, ϕ' are the structure morphisms of X and Y. In particular, u maps points of X lying over $s \in S$ to points of Y also lying over s. S-preschemes and S-morphisms form a category. We write $\operatorname{Hom}_S(X, Y)$ for the set of S-morphisms $X \to Y$. If $S = \operatorname{Spec}(A)$, we also use the term A-*morphism*.

(2.5.3). If X is an S-prescheme and $v\colon X' \to X$ is any morphism, the composite $X' \to X \to S$ makes X' an S-prescheme. In particular, open subschemes of an S-prescheme are naturally S-preschemes.

If $u\colon X \to Y$ is an S-morphism, then so is the restriction of u to any open $U \subseteq X$. Conversely, given an open covering $X = \bigcup_\alpha U_\alpha$, and S-morphisms $u_\alpha\colon U_\alpha \to Y$ which agree on every $U_\alpha \cap U_\beta$, there is a unique S-morphism $u\colon X \to Y$ such that every u_α is the restriction of u.

If U is an open subscheme of X, and V is an open subscheme of Y containing $u(U)$, then $u\colon U \to V$ is an S-morphism.

(2.5.4). Given a morphism $S' \to S$, the composite $X \to S' \to S$ makes any S'-prescheme an S-prescheme. Conversely, if S' is an open subscheme of S, and X is an S-prescheme such that the image of its structure morphism is contained in S', then X is also an S'-prescheme, and if Y is another S-prescheme with the same property, then any S-morphism $X \to Y$ is also an S'-morphism.

(2.5.5). An S-*section* of an S-prescheme X is an S-morphism $S \to X$ [Liu, 2.3.28]. We denote the set of S-sections of X by $\Gamma(X/S)$ [or by $X(S)$, as in Liu].